Ordinary Differential Equations - Part 22

$A \in \mathbb{R}^{n\times n} \implies e^{tA} \in \mathbb{R}^{n\times n}$ columns span solution space of $\dot{x} = Ax$

Remember: $\boxed{\begin{aligned}\dot{x} &= Ax\\ x(0) &= x_0\end{aligned}}$ has unique solution: $t \mapsto e^{tA} x_0$

Definition: $e^{tA} := \underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k}$ exists for every $t \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$

- each component is a function $t \in [a,b] \longrightarrow \mathbb{R}$
- we have uniform convergence

Properties: (a) derivative of the matrix exponential:

$$\frac{d}{dt} e^{tA} := \lim_{h\to 0} \frac{e^{(t+h)A} - e^{tA}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{\infty} \frac{(t+h)^k}{k!} A^k - \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{\infty} \frac{A^k}{k!} \left((t+h)^k - t^k \right) \right)$$

$$\overset{\text{uniform convergence}}{=} \sum_{k=0}^{\infty} \frac{A^k}{k!} \underbrace{\lim_{h\to 0} \frac{(t+h)^k - t^k}{h}}_{= \begin{cases} k \cdot t^{k-1} & , k \geq 1 \\ 0 & , k = 0 \end{cases}}$$

$$= \sum_{k=1}^{\infty} \frac{A^k}{k!} \cdot k \cdot t^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^{k-1} \cdot A = e^{tA} A = A e^{tA}$$

(b) exponentiation identity: $e^{A+B} = e^{A} e^{B}$ for matrices with $AB = BA$

(c) inverse:

$$\left.\begin{aligned} e^{A} e^{-A} &= e^{A-A} = e^{0} = \mathbb{1} \\ e^{-A} e^{A} &= e^{A-A} = e^{0} = \mathbb{1} \end{aligned}\right\} \quad (e^{A})^{-1} = e^{-A}$$