Ordinary Differential Equations - Part 21

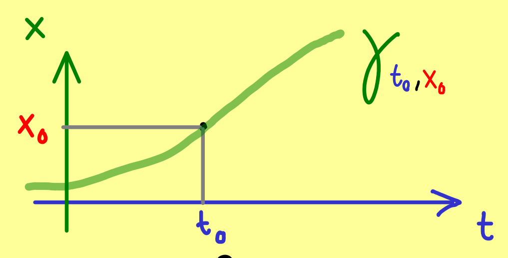
System of linear differential equations: $\dot{x} = A(t)\,x + b(t) \quad (*)$

with $I \ni t \longmapsto A(t) \in \mathbb{R}^{n\times n}$, $I \ni t \longmapsto b(t) \in \mathbb{R}^{n}$ continuous

interval in $\mathbb{R}$

We already know:

- the homogeneous part of $(*)$ $\left(\dot{x} = A(t)\,x\right)$ has an n-dimensional solution space S_0
- the initial value problem $(\mathrm{IVP}^{t_0}_{x_0})$ $\boxed{\begin{matrix}\dot{x} = A(t)\,x + b(t) \\ x(t_0) = x_0\end{matrix}}$ has a global solution $\gamma_{t_0,x_0} : I \longrightarrow \mathbb{R}^n$



Solution set: $S := \left\{ \beta : I \to \mathbb{R}^n \text{ continuously differentiable} \;\middle|\; \beta \text{ solution of } (*) \right\}$

$S_0 + \gamma_{t_0,x_0} := \left\{ \alpha + \gamma_{t_0,x_0} \;\middle|\; \alpha \in S_0 \right\}$ (affine subspace)

Show $S = S_0 + \gamma_{t_0,x_0}$:

$(\supseteq)$ Take $\alpha \in S_0$: $A(t)\left(\alpha(t) + \gamma_{t_0,x_0}(t)\right) + b(t)$

$$= \underbrace{A(t)\,\alpha(t)} + \underbrace{A(t)\,\gamma_{t_0,x_0}(t) + b(t)}$$

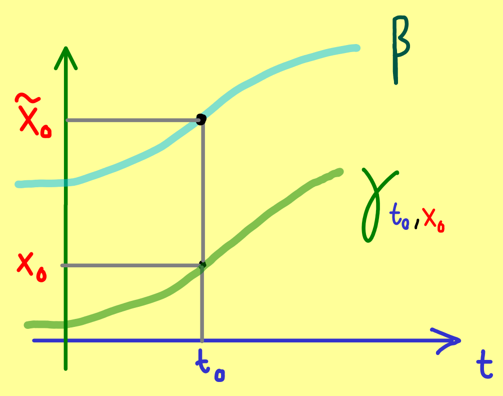
$$= \dot{\alpha}(t) + \dot{\gamma}_{t_0,x_0}(t)$$

$$= \left(\alpha + \gamma_{t_0,x_0}\right)^{\boldsymbol{\cdot}}(t)$$

$$\Longrightarrow \alpha + \gamma_{t_0,x_0} \in S$$

$(\subseteq)$ Take $\beta \in S$ and set $\tilde{x}_0 := \beta(t_0)$

$\Longrightarrow$ β is solution of $(\mathrm{IVP}^{t_0}_{\tilde{x}_0})$



Choose $\alpha \in S_0$ as the solution of the initial value problem $\boxed{\begin{matrix}\dot{x} = A(t)\,x \\ x(t_0) = \tilde{x}_0 - x_0\end{matrix}}$

Then: $\alpha + \gamma_{t_0,x_0} \in S$ with $\left(\alpha + \gamma_{t_0,x_0}\right)(t_0) = \alpha(t_0) + \gamma_{t_0,x_0}(t_0)$

$$= \tilde{x}_0 - x_0 + x_0 = \tilde{x}_0$$

$\Longrightarrow$ $\alpha + \gamma_{t_0,x_0}$ is solution of $(\mathrm{IVP}^{t_0}_{\tilde{x}_0})$

$\overset{\text{uniqueness}}{\Longrightarrow}$ $\beta = \alpha + \gamma_{t_0,x_0}$ $\square$

Result: The solution set of $\dot{x} = A(t)\,x + b(t)$ is given by

$$S = S_0 + \gamma$$

where S_0 is the solution space of the homogeneous part $\dot{x} = A(t)\,x$

and γ is a particular solution of $\dot{x} = A(t)\,x + b(t)$.

$\left(S \text{ is an n-dimensional affine subspace} \right)$