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ON STEADY

Ordinary Differential Equations — Part 11

$$\boxed{\begin{aligned}\dot{x} &= v(x)\\ x(0) &= x_0\end{aligned}}$$

initial value problem

with $v: \mathbb{R}^n \to \mathbb{R}^n$ loc. Lipschitz continuous

integrating

$$\underbrace{\int_0^t \dot{x}(s)\, ds}_{x(t) - x(0)} = \int_0^t v(x(s))\, ds$$

$$\Longrightarrow \quad x(t) = \underbrace{x_0 + \int_0^t v(x(s))\, ds}_{\Phi(x)}$$

function

<u>Definition:</u> $\Phi : \mathcal{F}(\mathbb{R}, \mathbb{R}^n) \longrightarrow \mathcal{F}(\mathbb{R}, \mathbb{R}^n)$

$$f \longmapsto \left(t \mapsto x_0 + \int_0^t v(f(s))\, ds \right)$$

<u>Now we know:</u> $x: \mathbb{R} \to \mathbb{R}^n$ is a solution of $\boxed{\begin{aligned}\dot{x} &= v(x)\\ x(0) &= x_0\end{aligned}}$

$$\Longleftrightarrow \quad \Phi(x) = x \qquad \text{(fixed point equation)}$$

<u>Banach fixed-point theorem:</u> Let (X, d) be a <u>complete</u> metric space (set with distance function)

and $\Phi: X \to X$ be a <u>contraction</u>, which means:

$$\exists\, q \in [0,1) \quad \forall x, \tilde{x} \in X: \quad d(\Phi(x), \Phi(\tilde{x})) \leq \underbrace{q}_{<1} \cdot d(x, \tilde{x}) .$$

Then: Φ has a unique fixed point $x^* \in X$ and

for each $x_0 \in X$ we have: $\Phi^n(x_0) \xrightarrow{n \to \infty} x^*$.