The Bright Side of Mathematics

Ordinary Differential Equations – Part 7

continuous functions

Linear ODE of first order: $\dot{x} = a(t)\cdot x + b(t)$

Finding solutions: (with an integrating factor)

$$\dot{x} + \tilde{a}(t)\,x = b(t) \qquad \text{with } \tilde{a}(t) := -a(t)$$

multiplying both sides

$$\Longleftrightarrow \quad \dot{x}\, e^{\tilde{A}(t)} + \tilde{a}(t)\, x\, e^{\tilde{A}(t)} = b(t)\, e^{\tilde{A}(t)}$$

Note: if $\tilde{A}$ is an antiderivative of $\tilde{a}$, then: $\frac{d}{dt} e^{\tilde{A}(t)} = \underbrace{\tilde{A}'(t)}_{=\tilde{a}(t)} \cdot e^{\tilde{A}(t)}$

product rule

$$\Longleftrightarrow \quad \frac{d}{dt}\left(x(t)\, e^{\tilde{A}(t)} \right) = \underbrace{b(t)\, e^{\tilde{A}(t)}}$$

H is antiderivative of $b(t)e^{\tilde{A}(t)}$

antiderivative

$$\Longleftrightarrow \quad x(t)\, e^{\tilde{A}(t)} = H(t) + c \, , \quad c \in \mathbb{R}$$

Solutions: $\alpha(t) = e^{-\tilde{A}(t)} \left(H(t) + c \right) , \quad c \in \mathbb{R}$

Example: $\dot{x} = t\,x + e^{\frac{1}{2}t^2} \, , \quad x(0) = x_0$

$$\Longleftrightarrow \quad \dot{x} - t\,x = e^{\frac{1}{2}t^2} \qquad \Big| \cdot e^{-\frac{1}{2}t^2}$$

$$\Longleftrightarrow \quad \dot{x} \cdot e^{-\frac{1}{2}t^2} - t\,x\,e^{-\frac{1}{2}t^2} = 1$$

$$\Longleftrightarrow \quad \frac{d}{dt}\left(x(t) \cdot e^{-\frac{1}{2}t^2} \right) = 1$$

$$\Longleftrightarrow \quad x(t) \cdot e^{-\frac{1}{2}t^2} = t + c \, , \quad c \in \mathbb{R}$$

$$\Longleftrightarrow \quad \text{solution: } \alpha(t) = (t + c) \cdot e^{\frac{1}{2}t^2}$$

Initial value condition: $\underbrace{\alpha(0)}_{= c} = x_0 \quad \rightsquigarrow \quad \alpha(t) = (t + x_0) \cdot e^{\frac{1}{2}t^2}$