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ON STEADY

The Bright Side of Mathematics

Ordinary Differential Equations – Part 5

Initial value problem: $\dot{x} = v(x)$ with $v: \mathbb{R} \to \mathbb{R}$ continuous

$x(0) = x_0$

Find all solutions $\alpha: (t_0, t_1) \to \mathbb{R}$ $\big(\dot{\alpha}(t) = v(\alpha(t))\big)$, $(t_0,t_1) \ni 0$

with $\alpha(0) = x_0$

Solving strategy: Assume $v(x_0) \neq 0$:

ODE: $\dfrac{\dot{x}}{v(x)} = 1$

Therefore: any solution $\alpha: (t_0, t_1) \to \mathbb{R}$ with $\alpha(0) = x_0$ satisfies: ($(t_0,t_1) \ni 0$)

fundamental theorem of calculus

$$\frac{\dot{\alpha}(s)}{v(\alpha(s))} = 1 \quad \text{for all } s \in (t_0, t_1)$$

$$\Longleftrightarrow \int_0^t \frac{\dot{\alpha}(s)}{v(\alpha(s))}\, ds = t \quad \text{for all } t \in (t_0, t_1)$$

substitution: $x = \alpha(s)$, $dx = \dot{\alpha}(s)\, ds$

$$\Longleftrightarrow \int_{x_0}^{\alpha(t)} \frac{1}{v(x)}\, dx = t \quad \text{for all } t \in (t_0, t_1)$$

$$\Longleftrightarrow F(\alpha(t)) - F(x_0) = t \quad \text{for all } t \in (t_0, t_1)$$

where F is an antiderivative of $\frac{1}{v}$

$$\Longleftrightarrow F(\alpha(t)) = t - c \quad \text{for all } t \in (t_0, t_1)$$

$$\Longleftrightarrow \alpha(t) = F^{-1}(t - c) \quad \text{for all } t \in (t_0, t_1)$$

Examples: (a) $\dot{x} = \lambda \cdot x$, $x(0) = x_0 \neq 0$

$$\Longleftrightarrow \frac{dx}{dt} = \lambda \cdot x \quad \overset{\text{informally}}{\Longleftrightarrow} \quad \int \frac{dx}{x} = \int \lambda \, dt$$

$$\Longleftrightarrow \log(|x|) = \lambda \cdot t + C \quad , \quad C \in \mathbb{R}$$

natural logarithm

$$\Longleftrightarrow |\alpha(t)| = e^{\lambda t} \cdot e^{C}$$

$$\Longleftrightarrow \alpha(t) = \begin{cases} -e^{C} e^{\lambda t} \\ e^{C} e^{\lambda t} \end{cases}$$

Solution: $\alpha(t) = x_0 \cdot e^{\lambda t}$

(b) $\dot{x} = x^2$, $x(0) = x_0 \neq 0$

$$\Longleftrightarrow \frac{dx}{dt} = x^2 \Longleftrightarrow \int \frac{dx}{x^2} = \int dt$$

$$\Longleftrightarrow -\frac{1}{x} = t + C \quad , \quad C \in \mathbb{R}$$

$$\Longleftrightarrow -\frac{1}{\alpha(t)} = t + C \quad , \quad C \in \mathbb{R}$$

$$\Longleftrightarrow \alpha(t) = \frac{-1}{t + C} \quad , \quad C \in \mathbb{R}$$

initial value: $\alpha(0) = \dfrac{-1}{C} \overset{!}{=} x_0 \Rightarrow \boxed{C = -\frac{1}{x_0}}$

solution: $\alpha(t) = \dfrac{-1}{t + (-\frac{1}{x_0})} = \dfrac{x_0}{1 - x_0 t}$