Ordinary Differential Equations – Part 12

initial value problem: $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}}$ with $v: \mathbb{R}^n \to \mathbb{R}^n$ loc. Lipschitz continuous

$\Longrightarrow$ there is a unique solution!

(Picard-Lindelöf theorem)

Banach fixed-point theorem: Let (X, d) be a complete metric space and $\Phi: X \to X$ be a contraction.

Then: Φ has a unique fixed point $x^* \in X$.

We need: (1) Complete metric space consisting of functions.

(2) Contraction $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\, ds$

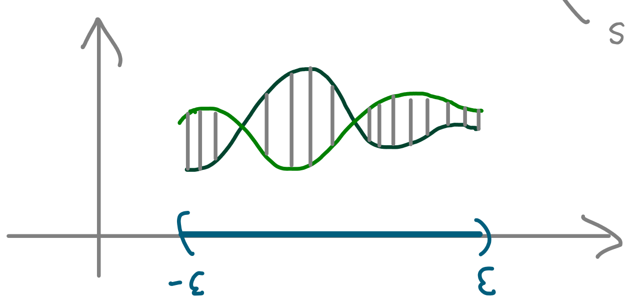
Now we know: $\alpha: \mathbb{R} \to \mathbb{R}^n$ is a solution of $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}}$

$\Longleftrightarrow \Phi(\alpha) = \alpha$ (fixed point equation)

For (1): $X = \left\{ \alpha: (-\varepsilon, \varepsilon) \to \tilde{U} \subseteq \mathbb{R}^n \text{ in the domain of } v \text{ with property } (*) \text{ (see below)} \;\middle|\; \alpha \text{ continuous + bounded}, \ \alpha(0) = x_0 \right\}$

with metric: $d(\alpha, \beta) := \sup_{t \in (-\varepsilon, \varepsilon)} \| \alpha(t) - \beta(t) \|_{\mathbb{R}^n}$

standard norm



Fact: (X, d) is a complete metric space.

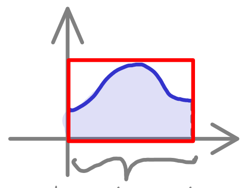
For (2): $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\, ds$ gives a map $\Phi : X \longrightarrow X$

$$d(\Phi(\alpha), \Phi(\beta)) = \sup_{t\in(-\varepsilon,\varepsilon)} \| \Phi(\alpha)(t) - \Phi(\beta)(t) \|_{\mathbb{R}^n}$$

$$= \sup_{t\in(-\varepsilon,\varepsilon)} \left\| \int_0^t \big(v(\alpha(s)) - v(\beta(s)) \big)\, ds \right\|_{\mathbb{R}^n}$$

triangle inequality for integrals

$$\leq \sup_{t\in(-\varepsilon,\varepsilon)} \int_0^t \| v(\alpha(s)) - v(\beta(s)) \|_{\mathbb{R}^n}\, ds$$



$$\leq \sup_{t\in(-\varepsilon,\varepsilon)} \underbrace{\text{length}([0,t])}_{|t| \leq \varepsilon} \cdot \underbrace{\sup_{s\in[0,t]} \| v(\alpha(s)) - v(\beta(s)) \|_{\mathbb{R}^n}}_{\leq \sup_{s\in(-\varepsilon,\varepsilon)} \cdots}$$

$$\leq \varepsilon \cdot \sup_{s\in(-\varepsilon,\varepsilon)} \underbrace{\| v(\alpha(s)) - v(\beta(s)) \|_{\mathbb{R}^n}}_{\leq L \, \| \alpha(s) - \beta(s) \|_{\mathbb{R}^n}}$$

(*) needed

$$\leq \underbrace{\varepsilon \cdot L}_{< 1 \text{ for } \varepsilon \text{ small enough}} \cdot d(\alpha, \beta)$$

contraction

Picard–Lindelöf theorem

$v: U \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous, $x_0 \in U$.

Then there is $\varepsilon > 0$ and a unique solution $\alpha : (-\varepsilon, \varepsilon) \longrightarrow U$

for the initial value problem $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}}$.

Definition of $\widetilde{U}$ with property $(*)$

V being locally Lipschitz continuous at x_0 means:

$$\underset{\delta>0}{\exists} \quad \underset{L\geq 0}{\exists} \quad \underset{y,z\in B_\delta(x)}{\forall} \; : \; \|v(\underset{\underset{\alpha(s)}{\uparrow}}{y}) - v(\underset{\underset{\beta(s)}{\uparrow}}{z})\| \leq L\cdot\|y-z\|$$

So we need $\alpha(s), \beta(s) \in B_\delta(x)$ for all $s\in(-\varepsilon,\varepsilon)$.

Hence: $\widetilde{U} := B_\delta(x)$ (not a problem for the solution since we choose ε as small as we want)