Ordinary Differential Equations - Part 12

initial value problem: $\boxed{\begin{aligned}\dot{x} &= v(x)\\ x(0) &= x_0\end{aligned}}$ with $v: \mathbb{R}^n \to \mathbb{R}^n$ loc. Lipschitz continuous

$\Longrightarrow$ there is a unique solution!

(Picard-Lindelöf theorem)

Banach fixed-point theorem: Let (X, d) be a complete metric space

and $\Phi : X \to X$ be a contraction.

Then: Φ has a unique fixed point $x^* \in X$.

We need: (1) Complete metric space consisting of functions.

(2) Contraction $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\, ds$

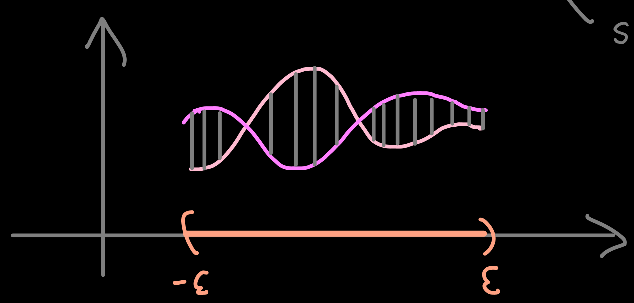
Now we know: $\alpha : \mathbb{R} \to \mathbb{R}^n$ is a solution of $\boxed{\begin{aligned}\dot{x} &= v(x)\\ x(0) &= x_0\end{aligned}}$

$\Longleftrightarrow \Phi(\alpha) = \alpha$ (fixed point equation)

For (1): $X = \left\{ \alpha : (-\varepsilon, \varepsilon) \to \tilde{U} \subseteq \mathbb{R}^n \right.$ in the domain of v with property $(*)$ (see below) $\left| \; \alpha \text{ continuous + bounded}, \; \alpha(0) = x_0 \right\}$

with metric: $d(\alpha, \beta) := \sup_{t \in (-\varepsilon, \varepsilon)} \| \alpha(t) - \beta(t) \|_{\mathbb{R}^n}$

standard norm



Fact: (X, d) is a complete metric space.

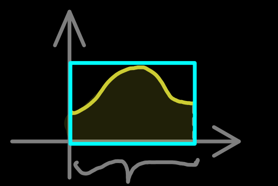
For (2): $\Phi(\alpha)(t) = x_0 + \int_0^t v(\alpha(s))\,ds$ gives a map $\Phi : X \longrightarrow X$

$$d(\Phi(\alpha),\Phi(\beta)) = \sup_{t\in(-\varepsilon,\varepsilon)} \left\| \Phi(\alpha)(t) - \Phi(\beta)(t) \right\|_{\mathbb{R}^n}$$

$$= \sup_{t\in(-\varepsilon,\varepsilon)} \left\| \int_0^t \big(v(\alpha(s)) - v(\beta(s)) \big)\, ds \right\|_{\mathbb{R}^n}$$

triangle inequality for integrals

$$\le \sup_{t\in(-\varepsilon,\varepsilon)} \int_0^t \left\| v(\alpha(s)) - v(\beta(s)) \right\|_{\mathbb{R}^n} ds$$



$$\le \sup_{t\in(-\varepsilon,\varepsilon)} \underbrace{\text{length}([0,t])}_{|t| \le \varepsilon} \cdot \underbrace{\sup_{s\in[0,t]} \left\| v(\alpha(s)) - v(\beta(s)) \right\|_{\mathbb{R}^n}}_{\le \sup_{s\in(-\varepsilon,\varepsilon)} \cdots}$$

$$\le \varepsilon \cdot \sup_{s\in(-\varepsilon,\varepsilon)} \underbrace{\left\| v(\alpha(s)) - v(\beta(s)) \right\|_{\mathbb{R}^n}}_{\le L \left\| \alpha(s) - \beta(s) \right\|_{\mathbb{R}^n}}$$

(*) needed

$$\le \underbrace{\varepsilon \cdot L}_{} \cdot d(\alpha,\beta)$$ contraction

< 1 for ε small enough

Picard-Lindelöf theorem

$v : U \longrightarrow \mathbb{R}^n$ loc. Lipschitz continuous, $x_0 \in U$.

Then there is $\varepsilon > 0$ and a unique solution $\alpha : (-\varepsilon,\varepsilon) \longrightarrow U$

for the initial value problem $\boxed{\begin{array}{l} \dot{x} = v(x) \\ x(0) = x_0 \end{array}}$.

Definition of $\widetilde{U}$ with property $(*)$

V being locally Lipschitz continuous at x_0 means:

$$\underset{\delta>0}{\exists} \quad \underset{L\geq 0}{\exists} \quad \underset{y,z\in B_\delta(x)}{\forall} \quad : \quad \| V(\underbrace{y}_{\alpha(s)}) - V(\underbrace{z}_{\beta(s)}) \| \leq L \cdot \| y - z \|$$

So we need $\alpha(s), \beta(s) \in B_\delta(x)$ for all $s \in (-\varepsilon, \varepsilon)$.

Hence: $\widetilde{U} := B_\delta(x)$ (not a problem for the solution since we choose ε as small as we want)