The Bright Side of Mathematics

Multivariable Calculus - Part 31

How does the method of Lagrange multipliers work?

There are C^1-functions given: $f: \mathbb{R}^n \to \mathbb{R}$, $g_j: \mathbb{R}^n \to \mathbb{R}$, $j \in \{1, \dots, m\}$

$\to$ $g: \mathbb{R}^n \to \mathbb{R}^m$

Then we search for points $x \in \mathbb{R}^n$ that satisfy $(n+m)$ equations:

$$g(x) = 0$$

$$\text{grad} f(x) = \sum_{j=1}^{m} \lambda_j \cdot \text{grad}\, g_j(x) \quad \text{for some } \lambda_1, \dots, \lambda_m \in \mathbb{R}$$

(Lagrange multipliers)

Definition: For C^1-functions $f: \mathbb{R}^n \to \mathbb{R}$, $g_j: \mathbb{R}^n \to \mathbb{R}$, we define

Lagrangian function: $L: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$L(x, \lambda) = f(x) - \lambda \cdot g(x)$$

(standard inner product in $\mathbb{R}^m$)

$$= f(x) - \lambda_1 \cdot g_1(x) - \lambda_2 \cdot g_2(x) - \dots - \lambda_m \cdot g_m(x)$$

Result: Gradient for $L: \mathbb{R}^{n+m} \to \mathbb{R}$

$$\text{grad}\, L(x, \lambda) = \begin{pmatrix} \frac{\partial L}{\partial x_1} \\ \vdots \\ \frac{\partial L}{\partial x_n} \\ \frac{\partial L}{\partial \lambda_1} \\ \vdots \\ \frac{\partial L}{\partial \lambda_m} \end{pmatrix} = \begin{pmatrix} \text{grad} f(x) - \sum_{j=1}^{m} \lambda_j \cdot \text{grad}\, g_j(x) \\ -g_1(x) \\ \vdots \\ -g_m(x) \end{pmatrix}$$

$$= \begin{pmatrix} \text{grad} f(x) - \sum_{j=1}^{m} \lambda_j \cdot \text{grad}\, g_j(x) \\ -g(x) \end{pmatrix}$$

Hence: $\text{grad}\, L(x, \lambda) = 0 \iff \left\{ \begin{array}{l} g(x) = 0 \\ \text{grad} f(x) = \sum_{j=1}^{m} \lambda_j \cdot \text{grad}\, g_j(x) \end{array} \right\}$

Method of Lagrange multipliers $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^m$ C^1-functions

f has a local extremum at $\tilde{x}$ subject to the constraint $g(x) = 0$, and $\text{rank}(J_g(\tilde{x})) = m$ $\implies$ $\text{grad}\, L(\tilde{x}, \lambda) = 0$ for some $\lambda_1, \lambda_2, \dots, \lambda_m$

Procedure:

- Form constraint $G = \{x \in \mathbb{R}^n \mid g(x) = 0\}$
- Check $\text{rank}(J_g(x)) = m$ for all $x \in G$
- Form $L(x, \lambda) = f(x) - \lambda \cdot g(x)$
- Solve $\text{grad}\, L(\tilde{x}, \lambda) = 0$ to get candidates $\tilde{x} \in \mathbb{R}^n$
- Use some more knowledge to decide if maximum/minimum