The Bright Side of Mathematics

Multivariable Calculus - Part 30

Method of Lagrange multipliers

$f: \mathbb{R}^n \to \mathbb{R}$, $g_j: \mathbb{R}^n \to \mathbb{R}$ C^1-functions for $j \in \{1, \dots, m\}$ → $g: \mathbb{R}^n \to \mathbb{R}^m$ C^1-function

f has a local extremum at $\tilde{x}$ subject to the constraint $g(x) = 0$, and $\text{rank}(J_g(\tilde{x})) = m$ $\Longrightarrow$ There are real numbers $\lambda_j \in \mathbb{R}$ (Lagrange multipliers):

$$\text{grad} f(\tilde{x}) = \sum_{j=1}^{m} \lambda_j \cdot \text{grad}\, g_j(\tilde{x})$$

Example: $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x_1, x_2, x_3) = 2 \cdot x_1 + 3 \cdot x_2 + 2 \cdot x_3$.

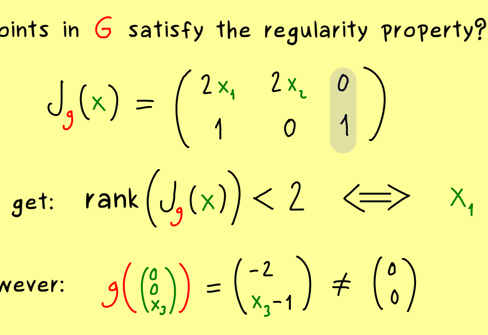
Search for extrema on the intersection of the cylinder $\{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 2 \}$ and the plane $\{ x \in \mathbb{R}^3 \mid x_1 + x_3 = 1 \}$.

(a) Formulate the contraints: $g: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 - 2 \\ x_1 + x_3 - 1 \end{pmatrix}$$

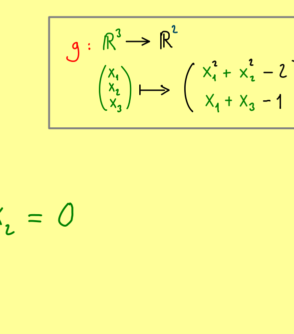
$g(x) = 0$ describes exactly this intersection!

(b) Let's sketch this!



$G := \{ x \in \mathbb{R}^3 \mid g(x) = 0 \}$

side view:



(c) Do all points in G satisfy the regularity property?

$$J_g(x) = \begin{pmatrix} 2x_1 & 2x_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$g: \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 - 2 \\ x_1 + x_3 - 1 \end{pmatrix}$

We get: $\text{rank}(J_g(x)) < 2 \iff x_1 = x_2 = 0$

However: $g\left(\begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ x_3 - 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Longrightarrow \text{rank}(J_g(x)) = 2$ for all $x \in G$

(d) Which points $x \in G$ satisfy the necessary condition for extrema?

$$\text{grad} f(x) = \lambda_1 \cdot \text{grad}\, g_1(x) + \lambda_2 \cdot \text{grad}\, g_2(x)$$

$$\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 2x_1 \\ 2x_2 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = 2 \cdot x_1 + 3 \cdot x_2 + 2 \cdot x_3$

$g: \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 - 2 \\ x_1 + x_3 - 1 \end{pmatrix}$

Five equations for five unknowns $x_1, x_2, x_3, \lambda_1, \lambda_2$:

(1) $x_1^2 + x_2^2 - 2 = 0$
(2) $x_1 + x_3 - 1 = 0$
(3) $2 = \lambda_1 \cdot 2x_1 + \lambda_2$
(4) $3 = \lambda_1 \cdot 2x_2$
(5) $2 = \lambda_2$

$\Longrightarrow$

(1) $x_1^2 + x_2^2 - 2 = 0$
(2) $x_1 + x_3 - 1 = 0$
(3) $0 = \lambda_1 \cdot 2x_1$ ← $x_1 = 0$ or $\lambda_1 = 0$ (conflict with (4))
(4) $3 = \lambda_1 \cdot 2x_2$
(5) $\lambda_2 = 2$

$\Longrightarrow$

(1) $x_2^2 = 2$
(2) $x_3 = 1$
(3) $x_1 = 0$
(4) $3 = \lambda_1 \cdot 2x_2$
(5) $\lambda_2 = 2$

$\Longrightarrow$ two candidates: $\tilde{x} = \begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}$, $\hat{x} = \begin{pmatrix} 0 \\ +\sqrt{2} \\ 1 \end{pmatrix}$

(e) Do we have extrema there?

$f(\tilde{x}) = -3\sqrt{2} + 2$ ← minimum

$f(\hat{x}) = 3\sqrt{2} + 2$ ← maximum

(continuous images of compact sets are compact!)