BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Multivariable Calculus - Part 27

Inverse Function Theorem: $U, V \subseteq \mathbb{R}^n$ open, $f \in C^k(U, V)$, $x_0 \in U$.

$$\det\left(J_f(x_0)\right) \neq 0 \Rightarrow f \text{ is a local } C^k\text{-diffeomorphism at } x_0$$

$(k \in \mathbb{N} \cup \{\infty\})$

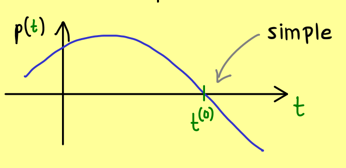
Implicit Function Theorem: $U \subseteq \mathbb{R}^k \times \mathbb{R}^m$ open, $F \in C^k(U, \mathbb{R}^m)$

Let $u^{(0)} = (\underbrace{x^{(0)}}_{\in \mathbb{R}^k}, \underbrace{y^{(0)}}_{\in \mathbb{R}^m}) \in U$ such that $F(u^{(0)}) = 0$.

If $\det\Big(\underbrace{\frac{\partial F}{\partial y}(u^{(0)})}_{\in \mathbb{R}^{m \times m}}\Big) \neq 0$, then there are open sets $V_1 \subseteq \mathbb{R}^k$, $V_2 \subseteq \mathbb{R}^m$ $\begin{pmatrix} x^{(0)} \in V_1 \\ y^{(0)} \in V_2 \end{pmatrix}$

and a map $g \in C^k(V_1, V_2)$ with $F(x, g(x)) = 0$ for all $x \in V_1$.

Application: Let's consider a polynomial $p(t) = a_0 + a_1 t + \dots + a_N t^N$, $N \in \mathbb{N}$



simple zero: $p(t^{(0)}) = 0$

$p'(t^{(0)}) = 0$

What happens if we change the coefficients a little bit?

For quadratic polynomial: $p(t) = a_0 + a_1 t + a_2 t^2$, $a_2 \neq 0$

finding zeros: $t^2 + \frac{a_1}{a_2} t + \frac{a_0}{a_2} = 0$

$$\Leftrightarrow t_\pm = \frac{-a_1 \pm \sqrt{a_1^2 - 4 a_2 a_0}}{2 a_2}$$

General case: $F : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$

$(x, t) \mapsto x_1 + x_2 t^1 + x_3 t^2 + \dots + x_{N+1} t^N$ $\quad k = N+1$

$\hookrightarrow$ C^∞-function $\quad F(x^{(0)}, t^{(0)}) = 0$, $\quad x^{(0)} := \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix}$

$$\frac{\partial F}{\partial t}(x^{(0)}, t^{(0)}) \neq 0$$

Implicit function theorem is applicable!

$\hookrightarrow$ we find a local inverse $g \in C^\infty(V_1, V_2)$

with $F(x, g(x)) = 0$ for all $x \in V_1$.