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ON STEADY

The Bright Side of
Mathematics

Multivariable Calculus – Part 26

Implicit function theorem: $U \subseteq \mathbb{R}^k \times \mathbb{R}^m$ open, $F \in C^1(U, \mathbb{R}^m)$

Let $u^{(0)} = (\underbrace{x^{(0)}}_{\in \mathbb{R}^k}, \underbrace{y^{(0)}}_{\in \mathbb{R}^m}) \in U$ such that $F(u^{(0)}) = 0$.

If $\det\Big(\underbrace{\frac{\partial F}{\partial y}(u^{(0)})}_{\in \mathbb{R}^{m \times m}}\Big) \neq 0$, then there are open sets $V_1 \subseteq \mathbb{R}^k$, $V_2 \subseteq \mathbb{R}^m$ $\begin{pmatrix} x^{(0)} \in V_1 \\ y^{(0)} \in V_2 \end{pmatrix}$

and a map $g \in C^1(V_1, V_2)$ with $F(x, g(x)) = 0$ for all $x \in V_1$.

Proof:

> **Inverse Function Theorem:** $U, V \subseteq \mathbb{R}^n$ (open), $f \in C^1(U, V)$, $u^{(0)} \in U$.
>
> $\det(J_f(u^{(0)})) \neq 0 \Rightarrow f$ is a local C^1-diffeomorphism at $u^{(0)}$

Define: $f: U \to \mathbb{R}^k \times \mathbb{R}^m$, $f(x,y) := \begin{pmatrix} x \\ F(x,y) \end{pmatrix}$

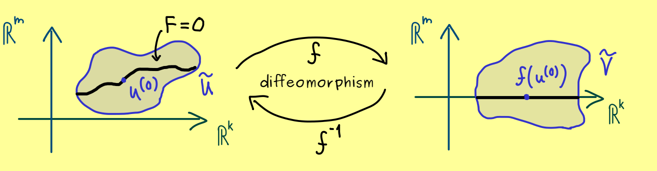
$\Rightarrow f \in C^1(U, \mathbb{R}^k \times \mathbb{R}^m)$

$$J_f(u^{(0)}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ | & | & & | \\ \frac{\partial F}{\partial x_1}(u^{(0)}) & \frac{\partial F}{\partial x_2}(u^{(0)}) & \cdots & \frac{\partial F}{\partial y_m}(u^{(0)}) \\ | & | & & | \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{1}_k & 0 \\ \hline \frac{\partial F}{\partial x}(u^{(0)}) & \frac{\partial F}{\partial y}(u^{(0)}) \end{array}\right)$$

Linear Algebra 49

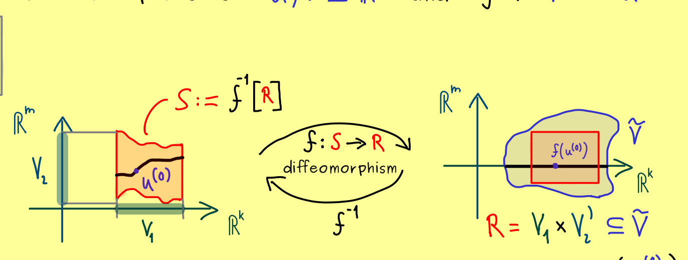
$$\Rightarrow \det(J_f(u^{(0)})) = \det(\mathbb{1}_k) \cdot \det\left(\frac{\partial F}{\partial y}(u^{(0)})\right) \neq 0$$

inverse function theorem $\Rightarrow$



We have open sets $\tilde{U}, \tilde{V} \subseteq \mathbb{R}^n$ and $f^{-1}: \tilde{V} \to \tilde{U}$ is C^1-function.

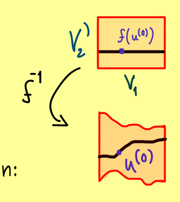
$f(x,y) = \begin{pmatrix} x \\ F(x,y) \end{pmatrix}$



$u^{(0)} \in S$ $\qquad$ $f(u^{(0)}) = \begin{pmatrix} x^{(0)} \\ 0 \end{pmatrix} \in R$

Consider map $f^{-1}: V_1 \times V_2' \longrightarrow V_1 \times V_2$

$\begin{pmatrix} x \\ y' \end{pmatrix} \longmapsto \begin{pmatrix} x \\ h(x, y') \end{pmatrix}$



$\Rightarrow$ second component of f^{-1} defines C^1-function:

$f(x,y) = \begin{pmatrix} x \\ F(x,y) \end{pmatrix}$

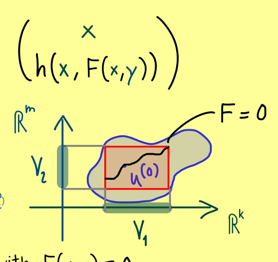
$h: V_1 \times V_2' \to V_2$

$g: V_1 \to V_2$, $g(x) = h(x, 0)$ — still C^1

$$\begin{pmatrix} x \\ y \end{pmatrix} = f^{-1}\left(f\begin{pmatrix} x \\ y \end{pmatrix}\right) = f^{-1}\begin{pmatrix} x \\ F(x,y) \end{pmatrix} = \begin{pmatrix} x \\ h(x, F(x,y)) \end{pmatrix}$$

$\Rightarrow y = h(x, F(x,y))$

for all $y \in V_2$, $x \in V_1$ (for which $F(x,y) \in V_2'$)



$F(x,y) = 0$

$\Rightarrow y = g(x)$ for all $x \in V_1$ with $F(x,y) = 0$

We have shown: If $(x,y) \in V_1 \times V_2$ with $F(x,y) = 0$, then $y = g(x)$.

Jacobian of g: $F(x, g(x)) = 0$ for all $x \in V_1$

$\quad \| $

$(F \circ G)(x)$ with $G(x) = \begin{pmatrix} x \\ g(x) \end{pmatrix}$

chain rule $\Rightarrow$ $0 = J_{F \circ G}(x) = J_F(G(x)) \, J_G(x)$

$$= \left(\frac{\partial F}{\partial x}(G(x)) \;\Big|\; \frac{\partial F}{\partial y}(G(x))\right) \begin{pmatrix} \mathbb{1}_k \\ \hline J_g(x) \end{pmatrix}$$

$$= \frac{\partial F}{\partial x}(G(x)) + \frac{\partial F}{\partial y}(G(x)) \, J_g(x) \qquad \square$$