BECOME A MEMBER

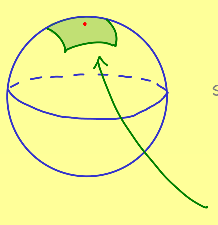
ON STEADY

The Bright Side of Mathematics

Multivariable Calculus – Part 25

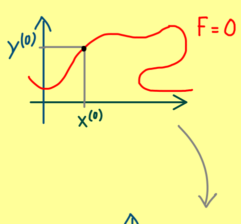
Recall: $x^2 + y^2 + z^2 = 1$

$\rightsquigarrow z(x,y) = \,?$



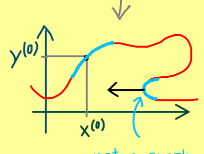
Implicit function theorem: $U \subseteq \mathbb{R}^k \times \mathbb{R}^m$ open, $F: U \to \mathbb{R}^m$ C^1-function.

Let $u^{(0)} = (\underbrace{x^{(0)}}_{\in \mathbb{R}^k}, \underbrace{y^{(0)}}_{\in \mathbb{R}^m}) \in U$ such that $F(u^{(0)}) = 0$



and
$$J_F(u^{(0)}) = \begin{pmatrix} \frac{\partial F}{\partial x_1}(u^{(0)}) \cdots \frac{\partial F}{\partial x_k}(u^{(0)}) & \frac{\partial F}{\partial y_1}(u^{(0)}) \cdots \frac{\partial F}{\partial y_m}(u^{(0)}) \end{pmatrix}$$

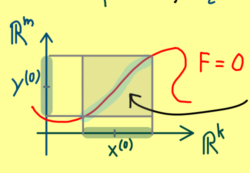
$$\frac{\partial F}{\partial x}(u^{(0)}) \in \mathbb{R}^{m \times k} \qquad \frac{\partial F}{\partial y}(u^{(0)}) \in \mathbb{R}^{m \times m}$$



If $\det\left(\frac{\partial F}{\partial y}(u^{(0)})\right) \neq 0$, then there are open sets $V_1 \subseteq \mathbb{R}^k$, $V_2 \subseteq \mathbb{R}^m$ $\begin{pmatrix} x^{(0)} \in V_1 \\ y^{(0)} \in V_2 \end{pmatrix}$

and a map $g \in C^1(V_1, V_2)$ with

$$F(x, g(x)) = 0 \quad \text{for all } x \in V_1 .$$



Moreover: If $(x,y) \in V_1 \times V_2$ with $F(x,y) = 0$, then $y = g(x)$.

For the derivative of g:
$$J_g(x) = -\left(\frac{\partial F}{\partial y}(x, g(x))\right)^{-1} \cdot \frac{\partial F}{\partial x}(x, g(x))$$
for all $x \in V_1$.