Multivariable Calculus – Part 24

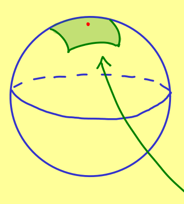
Inverse Function Theorem: $U, V \subseteq \mathbb{R}^n$ open, $f \in C^1(U,V)$, $x_0 \in U$.

$$\det\left(J_f(x_0)\right) \neq 0 \implies f \text{ is a local } C^1\text{-diffeomorphism at } x_0$$

Application: $x^2 + y^2 + z^2 = 1$

$\rightsquigarrow z(x,y) = \,?$

(explicit formula?)



sphere in $\mathbb{R}^3$

graph surface $z : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

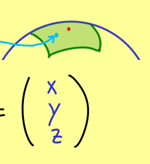
Let's take the point $(0,0,1)$:

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ x^2+y^2+z^2 \end{pmatrix}$$

$$J_f(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x & 2y & 2z \end{pmatrix} \implies \det\left(J_f(0,0,1)\right) = 2 \neq 0$$

inverse function theorem

$\implies f^{-1}$ exists locally as a C^1-function



$$f^{-1}\begin{pmatrix} x \\ y \\ x^2+y^2+z^2 \end{pmatrix} = f^{-1}\left(f\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightsquigarrow f^{-1}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\implies z(x,y) := \text{last component of } f^{-1}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Here we know: $z(x,y) = \sqrt{1-x^2-y^2}$ around $(0,0,1)$