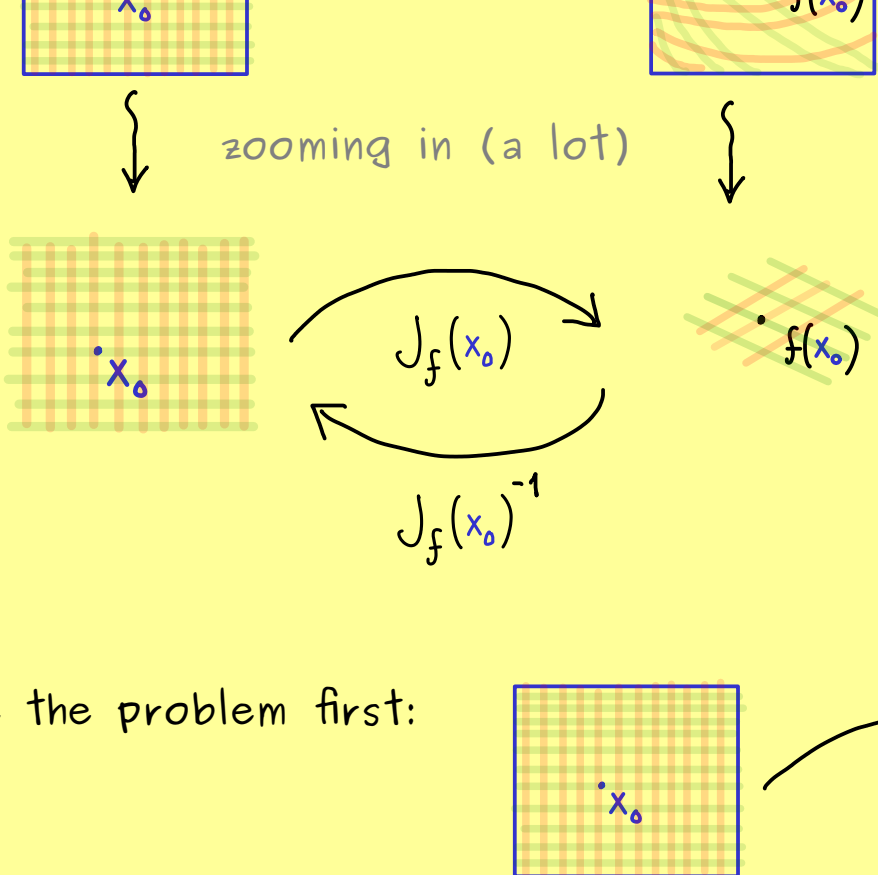


Multivariable Calculus - Part 23

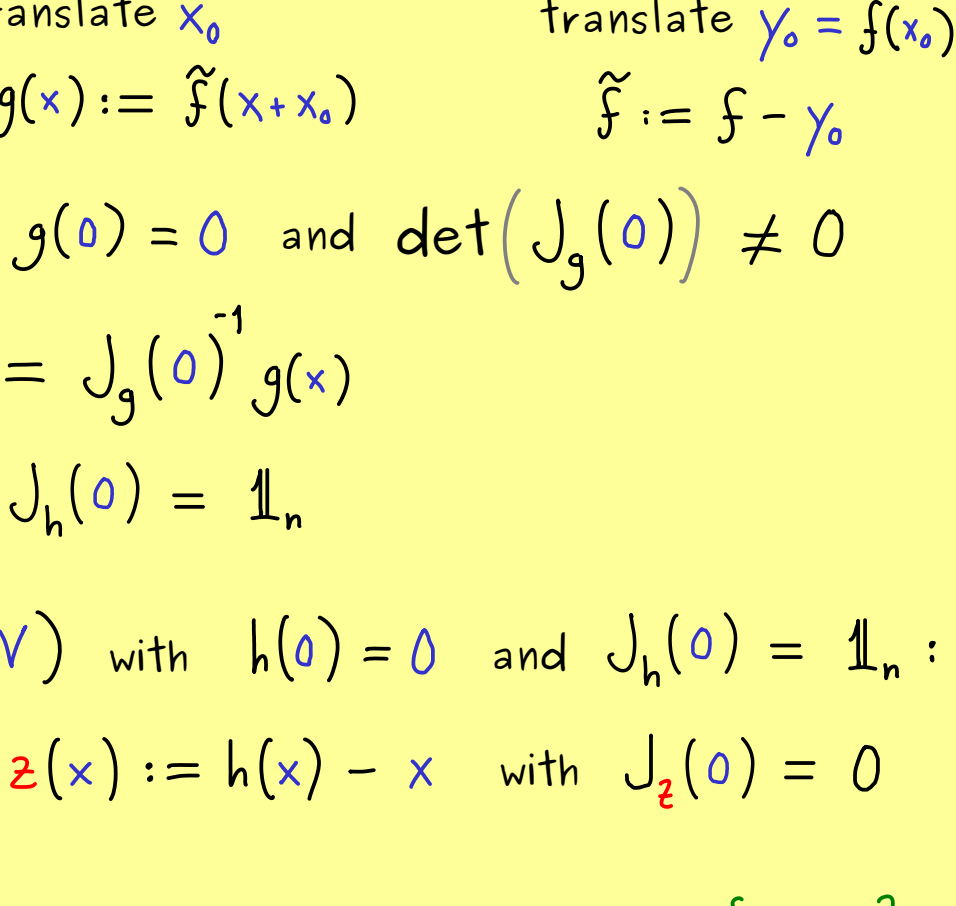
Inverse Function Theorem: $U, V \subseteq \mathbb{R}^n$, $f \in C^1(U, V)$, $x_0 \in U$.

$$\det(J_f(x_0)) \neq 0 \implies f \text{ is a local } C^1\text{-diffeomorphism at } x_0$$

Visualization:

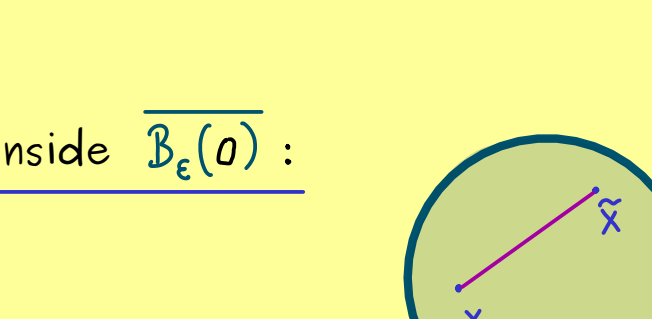


Proof: Let's normalize the problem first:



Let's prove the theorem for $h \in C^1(U, V)$ with $h(0) = 0$ and $J_h(0) = \mathbb{1}_n$:

Let's compare h to the identity map: $z(x) := h(x) - x$ with $J_z(0) = 0$

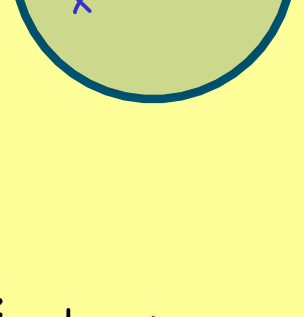


Choose $\epsilon > 0$ such that for all $i, j \in \{1, \dots, n\}$

$$\left| \frac{\partial z_i}{\partial x_j}(x) \right| \leq \frac{1}{2} \cdot \frac{1}{n^2} \text{ for all } x \in \overline{B_\epsilon(0)}$$

and $\det(J_h(x)) \neq 0$ for all $x \in \overline{B_\epsilon(0)}$.

Inside $\overline{B_\epsilon(0)}$:



$$a(t) := z_i(x + t(\tilde{x} - x)), \quad t \in [0, 1]$$

mean value theorem $\implies a(1) - a(0) = a'(\xi) \cdot (1-0)$

chain rule $= \langle \text{grad } z_i(x + \xi(\tilde{x} - x)), \tilde{x} - x \rangle$

Hence: $|z_i(\tilde{x}) - z_i(x)|^2 \leq \|\text{grad } z_i(x + \xi(\tilde{x} - x))\|^2 \|\tilde{x} - x\|^2$

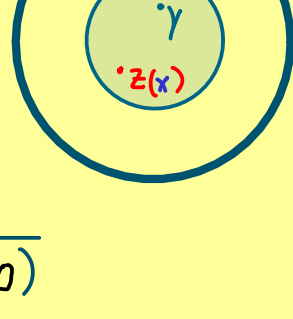
$$\leq n \cdot \frac{1}{4} \cdot \frac{1}{n^4} \|\tilde{x} - x\|^2 \leq \frac{1}{4} \cdot \frac{1}{n^2} \|\tilde{x} - x\|^2$$

$$\implies \|z(\tilde{x}) - z(x)\|^2 = \sum_{i=1}^n |z_i(\tilde{x}) - z_i(x)|^2 \leq n \cdot \frac{1}{4} \cdot \frac{1}{n^2} \|\tilde{x} - x\|^2 = \frac{1}{4} \cdot \frac{1}{n^2} \|\tilde{x} - x\|^2$$

$$\implies \|z(\tilde{x}) - z(x)\| \leq \frac{1}{2n} \|\tilde{x} - x\| \leq \frac{1}{2} \|\tilde{x} - x\|$$

$\hookrightarrow z$ is contraction

Let's fix an element $y \in \overline{B_{\epsilon/2}(0)}$:



$$z^{[y]}: \overline{B_{\epsilon/2}(0)} \rightarrow \overline{B_{\epsilon/2}(0)}$$

$$z^{[y]}(x) := y - z(x)$$

$$\implies \|z^{[y]}(\tilde{x}) - z^{[y]}(x)\| \leq \frac{1}{2} \|\tilde{x} - x\|$$

contraction:

Banach fixed-point theorem

$$\implies \text{there is exactly one fixed point } x^* : z^{[y]}(x^*) = x^*$$

Note: $z^{[y]}(x) = x \iff y - z(x) = x \iff y - (h(x) - x) = x \iff y = h(x)$

Result: For each $y \in \overline{B_{\epsilon/2}(0)}$, there is exactly one $x^* \in \overline{B_\epsilon(0)}$ with

$$h(x^*) = y.$$

$$\implies h \text{ is invertible as a map } h: \tilde{U} \rightarrow \tilde{V} = \overline{B_{\epsilon/2}(0)}$$

$$h^{-1}[\tilde{V}]$$

$$\implies h^{-1}: \tilde{V} \rightarrow \tilde{U} \text{ is well-defined}$$

- To show:
- h^{-1} is continuous
 - h^{-1} is differentiable
 - $h^{-1} \in C^1(\tilde{V}, \tilde{U})$

(1) $x, \tilde{x} \in U$: $\|\tilde{x} - x\| = \|h(\tilde{x}) - z(\tilde{x}) - h(x) + z(x)\|$

$$z(x) := h(x) - x \leq \underbrace{\|h(\tilde{x}) - h(x)\|}_{\tilde{y}} + \underbrace{\|z(\tilde{x}) - z(x)\|}_{\leq \frac{1}{2} \|\tilde{x} - x\|}$$

$$\implies \frac{1}{2} \|\tilde{x} - x\| \leq \|\tilde{y} - y\|$$

$$\implies \|h^{-1}(y) - h^{-1}(\tilde{y})\| \leq 2 \|\tilde{y} - y\| \implies h^{-1} \text{ is continuous}$$

(2) h is totally differentiable at x :

$$h(x) - h(\tilde{x}) = J_h(x)(x - \tilde{x}) + \phi(x - \tilde{x}) \text{ with } \frac{\|\phi(x - \tilde{x})\|}{\|x - \tilde{x}\|} \xrightarrow{\tilde{x} \rightarrow x} 0$$

$$\implies J_h(x)^{-1}(y - \tilde{y}) + J_h(x)^{-1}(-1)\phi(x - \tilde{x}) = h^{-1}(y) - h^{-1}(\tilde{y})$$

$$J_{h^{-1}}(y) \text{ satisfies: } \frac{\gamma(y - \tilde{y})}{\|y - \tilde{y}\|} \xrightarrow{\tilde{y} \rightarrow y} 0$$

(3) $J_{h^{-1}}(y) = J_h(h^{-1}(y))^{-1}$ \leftarrow composition of continuous functions □