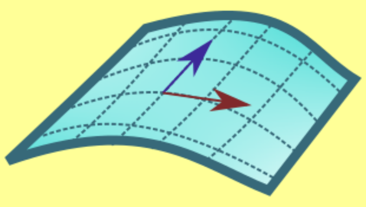
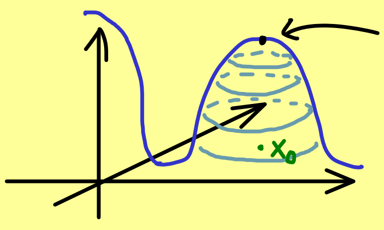
BECOME A MEMBER

ON STEADY

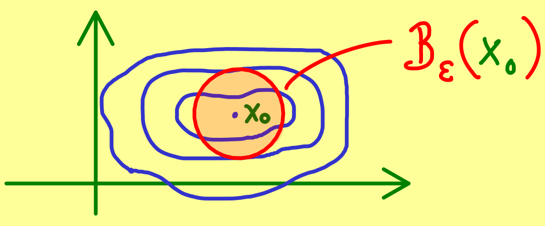
The Bright Side of
Mathematics



Multivariable Calculus – Part 18



local extremum
for $f: \mathbb{R}^2 \to \mathbb{R}$



Definition: $\mathcal{D} \subseteq \mathbb{R}^n$, $f: \mathcal{D} \to \mathbb{R}$.

(a) f has a local maximum at $x_0 \in \mathcal{D}$ if there is an $\varepsilon > 0$ such that

$$f(x_0) \geq f(x) \quad \text{for all } x \in \mathcal{D} \cap \mathcal{B}_\varepsilon(x_0).$$

(b) f has an isolated local maximum at $x_0 \in \mathcal{D}$ if there is an $\varepsilon > 0$ such that

$$f(x_0) > f(x) \quad \text{for all } x \in \mathcal{D} \cap \mathcal{B}_\varepsilon(x_0).$$

(c) f has a local minimum at $x_0 \in \mathcal{D}$ if there is an $\varepsilon > 0$ such that

$$f(x_0) \leq f(x) \quad \text{for all } x \in \mathcal{D} \cap \mathcal{B}_\varepsilon(x_0).$$

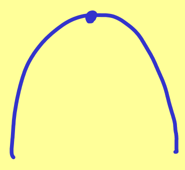
(d) f has an isolated local minimum at $x_0 \in \mathcal{D}$ if there is an $\varepsilon > 0$ such that

$$f(x_0) < f(x) \quad \text{for all } x \in \mathcal{D} \cap \mathcal{B}_\varepsilon(x_0).$$

(e) f has a local extremum at $x_0 \in \mathcal{D}$ if f has a local maximum or local minimum at $x_0 \in \mathcal{D}$

Necessary condition: Let $f \in C^1(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$.

f has a local extremum at x_0 $\Rightarrow$ $\operatorname{grad} f(x_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$



Sufficient condition: Let $f \in C^3(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ be a critical point $\left(\operatorname{grad} f(x_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$.

Then: $f(x_0 + h) = f(x_0) + \frac{1}{2} h^T H_f(x_0) h + \psi(h)$ and:

(1) $H_f(x_0)$ positive definite $\Rightarrow$ f has an isolated local minimum at x_0

$\left(h^T H_f(x_0) h > 0 \text{ for all } h \neq 0 \right)$

(2) $H_f(x_0)$ negative definite $\Rightarrow$ f has an isolated local maximum at x_0

$\left(h^T H_f(x_0) h < 0 \text{ for all } h \neq 0 \right)$

(3) $H_f(x_0)$ indefinite $\Rightarrow$ f has not a local extremum at x_0

(saddle point)

$\left(\begin{array}{l} \text{There is } h^T H_f(x_0) h < 0 \\ \text{and } \tilde{h}^T H_f(x_0) \tilde{h} > 0 \end{array} \right)$

$$\boxed{f(x_0 + h) = f(x_0) + \tfrac{1}{2} h^T H_f(x_0) h + \psi(h)}$$

(4) f has a local maximum at x_0 $\Rightarrow$ $H_f(x_0)$ negative semi-definite

$\left(h^T H_f(x_0) h \leq 0 \text{ for all } h \neq 0 \right)$

(5) f has a local minimum at x_0 $\Rightarrow$ $H_f(x_0)$ positive semi-definite

$\left(h^T H_f(x_0) h \geq 0 \text{ for all } h \neq 0 \right)$