Multivariable Calculus – Part 17

Taylor's theorem: $f \in C^\infty(\mathbb{R}^n)$, $\tilde{x} \in \mathbb{R}^n$

k-th order Taylor polynomial: $$T_k(h) = \sum_{|\alpha| \le k} \frac{D^\alpha f(\tilde{x})}{\alpha!} \cdot h^\alpha$$

remainder term: $$R_k(h) = \sum_{|\alpha| = k+1} \frac{D^\alpha f(\xi)}{\alpha!} \cdot h^\alpha$$

General second order Taylor polynomial:

$$T_2(h) = \sum_{|\alpha| \le 2} \frac{D^\alpha f(\tilde{x})}{\alpha!} \cdot h^\alpha = f(\tilde{x}) + \sum_{|\alpha| = 1} \frac{D^\alpha f(\tilde{x})}{\alpha!} \cdot h^\alpha + \sum_{|\alpha| = 2} \frac{D^\alpha f(\tilde{x})}{\alpha!} \cdot h^\alpha$$

$\alpha = (0,\dots,0,1,0..0)$

$$= f(\tilde{x}) + \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\tilde{x}) \cdot h_j}_{J_f(\tilde{x})\, h} + \sum_{|\alpha| = 2} \frac{D^\alpha f(\tilde{x})}{\alpha!} \cdot h^\alpha$$

$\alpha = (0,\dots,0,2,0..0)$, $\alpha! = 2$
or
$\alpha = (0,\dots,0,1,0..0,1,0,\dots,0)$

$$= f(\tilde{x}) + J_f(\tilde{x})\, h + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}(\tilde{x})\, h_j^2 + \frac{1}{2} \cdot \sum_{j \neq i} \frac{\partial^2 f}{\partial x_j \partial x_i}(\tilde{x})\, h_j h_i$$

$$= f(\tilde{x}) + J_f(\tilde{x})\, h + \frac{1}{2} \sum_{i,j} h_j \frac{\partial^2 f}{\partial x_j \partial x_i}(\tilde{x})\, h_i$$

$$= f(\tilde{x}) + J_f(\tilde{x})\, h + \frac{1}{2} h^T \underbrace{H_f(\tilde{x})}_{\text{Hessian matrix}} h$$

$$\big(H_f(\tilde{x})\big)_{ji} = \frac{\partial^2 f}{\partial x_j \partial x_i}(\tilde{x})$$

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x_1, x_2) = \cos(x_1 \cdot x_2)$

$$J_f(\tilde{x}) = \begin{pmatrix} -\tilde{x}_2 \sin(\tilde{x}_1 \cdot \tilde{x}_2) & -\tilde{x}_1 \sin(\tilde{x}_1 \cdot \tilde{x}_2) \end{pmatrix}$$

$$H_f(\tilde{x}) = \begin{pmatrix} -\tilde{x}_2^2 \cos(\tilde{x}_1 \cdot \tilde{x}_2) & -\sin(\tilde{x}_1 \cdot \tilde{x}_2) - \tilde{x}_2 \tilde{x}_1 \cos(\tilde{x}_1 \cdot \tilde{x}_2) \\ -\sin(\tilde{x}_1 \cdot \tilde{x}_2) - \tilde{x}_2 \tilde{x}_1 \cos(\tilde{x}_1 \cdot \tilde{x}_2) & -\tilde{x}_1^2 \cos(\tilde{x}_1 \cdot \tilde{x}_2) \end{pmatrix}$$

Taylor polynomial for expansion point $\tilde{x} = (0,0)$: $J_f(\tilde{x}) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $H_f(\tilde{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$T_2(h) = 1$$

Use $\cos(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} t^{2j}$: $T_k(h) = \sum_{j=0}^{m} \frac{(-1)^j}{(2j)!} \left(h_1^{2j} \cdot h_2^{2j}\right)$

order $k = 4 \cdot m$