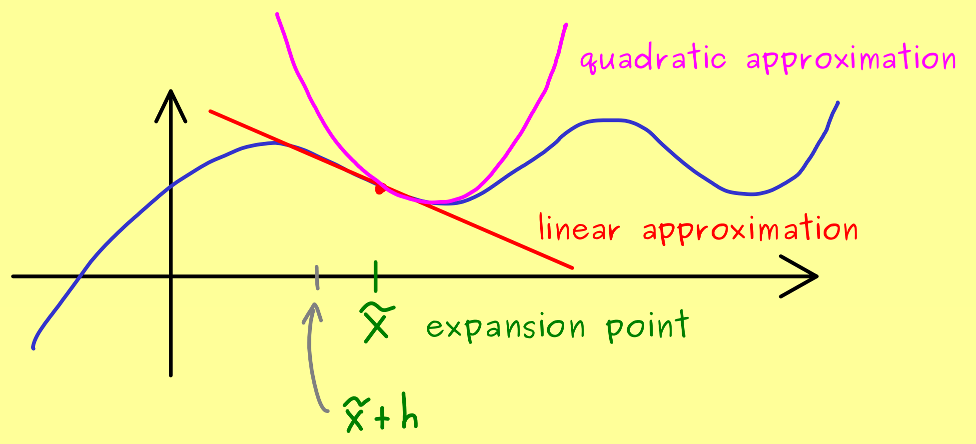
Multivariable Calculus – Part 16

Taylor's theorem:



$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

linear approximation: $f(\tilde{x}+h) = f(\tilde{x}) + J_f(\tilde{x})\, h + \phi(h)$ with $\dfrac{\phi(h)}{\|h\|} \xrightarrow{h \to 0} 0$

quadratic approximation: $f(\tilde{x}+h) = f(\tilde{x}) + J_f(\tilde{x})\, h + \frac{1}{2} h^T \underbrace{H_f(\tilde{x})}_{\text{Hessian matrix}}\, h + \psi(h)$

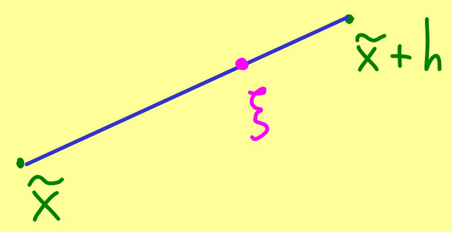
with $\dfrac{\psi(h)}{\|h\|^2} \xrightarrow{h \to 0} 0$

Theorem: Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that all $(k+1)$th-order partial derivative exist and form continuous functions $\left(f \in C^{k+1}(\mathbb{R}^n)\right)$.

Then: (for all $\tilde{x}, h \in \mathbb{R}^n$)

$$f(\tilde{x}+h) = \underbrace{T_k(h)}_{k\text{-th order Taylor polynomial}} + \overbrace{R_k(h)}^{\text{remainder term}}$$

$$= \overbrace{\sum_{|\alpha| \leq k} \frac{D^\alpha f(\tilde{x})}{\alpha!} \cdot h^\alpha} + \sum_{|\alpha| = k+1} \frac{D^\alpha f(\xi)}{\alpha!} \cdot h^\alpha$$



where ξ lies on the line between $\tilde{x}$ and $\tilde{x}+h$