Multivariable Calculus – Part 13

Schwarz's theorem (symmetry of second derivatives)

$U \subseteq \mathbb{R}^n$ open, $f: U \rightarrow \mathbb{R}$.

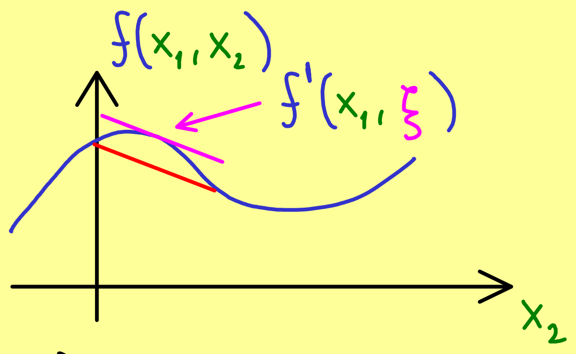
If all second-order partial derivatives exist (at all points $\tilde{x} \in U$)

and they form continuous functions $U \longrightarrow \mathbb{R}$,

then:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\tilde{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\tilde{x}) \quad \text{for all } i,j \text{ and } \tilde{x} \in U$$

Proof: For $n=2$, $i=1$, $j=2$, $\tilde{x}=0$ Idea:



$$\underbrace{f(h_1,h_2) - f(h_1,0)}_{u(h_1,h_2)} \underbrace{- f(0,h_2) + f(0,0)}_{-u(0,h_2)}$$

$$\overset{\text{mean value theorem}}{=} h_1 \frac{\partial u}{\partial x_1}(\xi_1,h_2) = h_1\left(\frac{\partial f}{\partial x_1}(\xi_1,h_2) - \frac{\partial f}{\partial x_1}(\xi_1,0)\right)$$

$$\overset{\text{mean value theorem}}{=} h_1 \cdot h_2 \cdot \frac{\partial^2 f}{\partial x_2 \partial x_1}(\xi_1,\xi_2) \quad \text{with } \xi_i \text{ between } 0 \text{ and } h_i$$

or:

$$\underbrace{f(h_1,h_2) - f(0,h_2)}_{v(h_1,h_2)} \underbrace{- f(h_1,0) + f(0,0)}_{-v(h_1,0)}$$

$$\overset{\text{mean value theorem}}{=} h_2 \frac{\partial v}{\partial x_2}(h_1,\eta_2) = h_2\left(\frac{\partial f}{\partial x_2}(h_1,\eta_2) - \frac{\partial f}{\partial x_2}(0,\eta_2)\right)$$

$$\overset{\text{mean value theorem}}{=} h_2 \cdot h_1 \frac{\partial^2 f}{\partial x_1 \partial x_2}(\eta_1,\eta_2) \quad \text{with } \eta_i \text{ between } 0 \text{ and } h_i$$

$$\Longrightarrow \frac{\partial^2 f}{\partial x_2 \partial x_1}(\xi_1,\xi_2) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(\eta_1,\eta_2)$$

$$\xRightarrow[\text{continuity}]{h_1,h_2 \rightarrow 0} \frac{\partial^2 f}{\partial x_2 \partial x_1}(0,0) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0) \qquad \square$$