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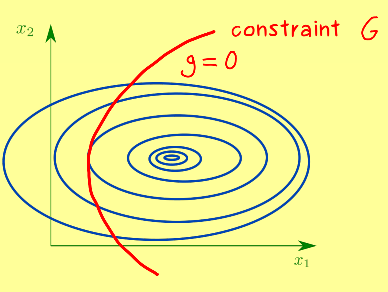
ON STEADY

Multivariable Calculus - Part 29

Method of Lagrange multipliers in $\mathbb{R}^2$: $f: \mathbb{R}^2 \to \mathbb{R}$, $g: \mathbb{R}^2 \to \mathbb{R}$ C^1-functions.

(2) f has a local extremum at $\tilde{x}$ subject to the constraint $g(x) = 0$,

and $\underbrace{\text{grad}\, g(\tilde{x}) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{(1)}$

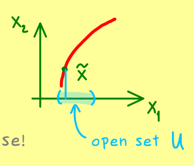


$\Longrightarrow$ There is a real number $\lambda \in \mathbb{R}$: ← Lagrange multiplier

$$\text{grad}\, f(\tilde{x}) = \lambda \cdot \text{grad}\, g(\tilde{x})$$

Proof: (1) $\Longrightarrow$ $g(x_1, x_2) = 0$ can be written locally as $x_1 = \beta(x_2)$ or $x_2 = \gamma(x_1)$ (assume this case!)

(↑ implicit function theorem)



$g(x_1, \gamma(x_1)) = 0$ for all $x_1 \in U$ (γ: C^1-function)

$$\Longrightarrow \frac{d}{dx_1} g(x_1, \gamma(x_1)) = 0$$

|| chain rule

$$J_g(x_1, \gamma(x_1)) \begin{pmatrix} 1 \\ \gamma'(x_1) \end{pmatrix} = \left\langle \text{grad}\, g(x_1, \gamma(x_1)), \begin{pmatrix} 1 \\ \gamma'(x_1) \end{pmatrix} \right\rangle$$

for all $x_1 \in U$ (standard inner product)

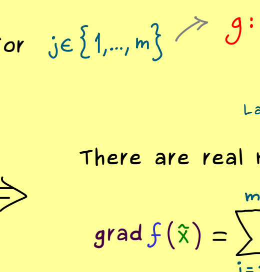
(2) $\Longrightarrow$ the function $\hat{f}: U \to \mathbb{R}$, $\hat{f}(x_1) = f(x_1, \gamma(x_1))$

has a local extremum at $\tilde{x}_1$ (where $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$)

$$\Longrightarrow 0 = \hat{f}'(\tilde{x}_1) = \left\langle \text{grad}\, f(\underbrace{\tilde{x}_1, \gamma(\tilde{x}_1)}_{\tilde{x}}), \begin{pmatrix} 1 \\ \gamma'(\tilde{x}_1) \end{pmatrix} \right\rangle$$

(standard inner product)

In summary: $\text{grad}\, f(\tilde{x})$ and $\text{grad}\, g(\tilde{x})$ are orthogonal to $\begin{pmatrix} 1 \\ \gamma'(\tilde{x}_1) \end{pmatrix}$



$\Longrightarrow$ There is a real number $\lambda \in \mathbb{R}$:

$$\text{grad}\, f(\tilde{x}) = \lambda \cdot \text{grad}\, g(\tilde{x})$$ □

Method of Lagrange multipliers (general version):

$f: \mathbb{R}^n \to \mathbb{R}$, $g_j: \mathbb{R}^n \to \mathbb{R}$ C^1-functions for $j \in \{1, \dots, m\}$ ↗ $g: \mathbb{R}^n \to \mathbb{R}^m$ C^1-function

f has a local extremum at $\tilde{x}$ subject to the constraint $g(x) = 0$,

and $\text{rank}\left(J_g(\tilde{x})\right) = m$

$\Longrightarrow$ There are real numbers $\lambda_j \in \mathbb{R}$ (Lagrange multipliers):

$$\text{grad}\, f(\tilde{x}) = \sum_{j=1}^{m} \lambda_j \cdot \text{grad}\, g_j(\tilde{x})$$