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ON STEADY

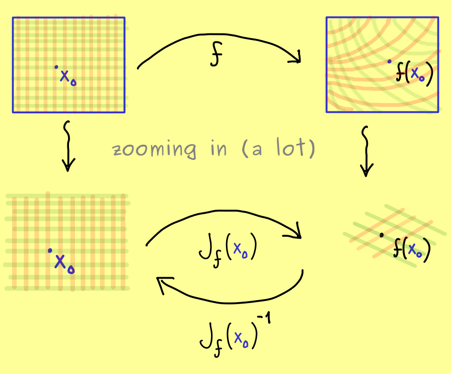
The Bright Side of
Mathematics

Multivariable Calculus – Part 23

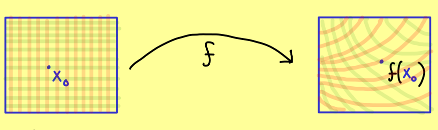
Inverse Function Theorem: $U, V \subseteq \mathbb{R}^n$ (open), $f \in C^1(U,V)$, $x_0 \in U$.

$$\det\big(J_f(x_0)\big) \neq 0 \Rightarrow f \text{ is a local } C^1\text{-diffeomorphism at } x_0$$

Visualization:



Proof: Let's normalize the problem first:



translate x_0: $g(x) := \tilde{f}(x + x_0)$

translate $y_0 = f(x_0)$: $\tilde{f} := f - y_0$

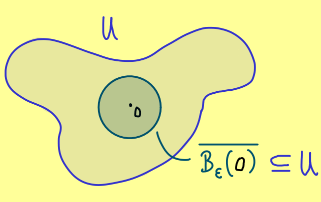
$\rightsquigarrow$ $g(0) = 0$ and $\det\big(J_g(0)\big) \neq 0$

Define: $h(x) := J_g(0)^{-1} g(x)$

$$\rightsquigarrow J_h(0) = \mathbb{1}_n$$

Let's prove the theorem for $h \in C^1(U,V)$ with $h(0) = 0$ and $J_h(0) = \mathbb{1}_n$:

Let's compare h to the identity map: $z(x) := h(x) - x$ with $J_z(0) = 0$

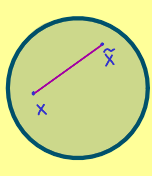


Choose $\varepsilon > 0$ such that for all $i,j \in \{1,\dots,n\}$

$$\left|\frac{\partial z_i}{\partial x_j}(x)\right| \leq \frac{1}{2}\cdot\frac{1}{n^2} \quad \text{for all } x \in \overline{B_\varepsilon(0)}$$

and $\det\big(J_h(x)\big) \neq 0$ for all $x \in \overline{B_\varepsilon(0)}$.

Inside $\overline{B_\varepsilon(0)}$:



$a(t) := z_i\big(x + t(\tilde{x} - x)\big)$, $t \in [0,1]$

mean value theorem:

$$a(1) - a(0) = a'(\xi)\cdot(1-0)$$

$$\overset{\text{chain rule}}{=} \left\langle \operatorname{grad} z_i\big(x + \xi(\tilde{x}-x)\big), \tilde{x} - x \right\rangle$$

Hence:

$$\big|z_i(\tilde{x}) - z_i(x)\big|^2 \leq \underbrace{\big\|\operatorname{grad} z_i\big(x+\xi(\tilde{x}-x)\big)\big\|^2}_{\sum_j \left|\frac{\partial z_i}{\partial x_j}(\cdot)\right|^2 \leq n\cdot\frac{1}{4}\cdot\frac{1}{n^4}} \cdot \|\tilde{x} - x\|^2$$

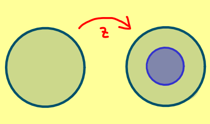
(with $\left|\frac{\partial z_i}{\partial x_j}(\cdot)\right| \leq \frac{1}{2}\cdot\frac{1}{n^2}$)

$$\Rightarrow \|z(\tilde{x}) - z(x)\|^2 = \sum_{i=1}^{n} \big|z_i(\tilde{x}) - z_i(x)\big|^2$$

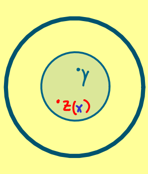
$$\leq n\cdot n\cdot\frac{1}{4}\cdot\frac{1}{n^4}\, \|\tilde{x} - x\| = \frac{1}{4}\cdot\frac{1}{n^2}\|\tilde{x} - x\|$$

$$\Rightarrow \|z(\tilde{x}) - z(x)\| \leq \frac{1}{2n}\|\tilde{x} - x\| \leq \frac{1}{2}\|\tilde{x} - x\|$$

$\hookrightarrow$ z is contraction



Let's fix an element $y \in \overline{B_{\frac{\varepsilon}{2}}(0)}$:



$z^{[y]} : \overline{B_\varepsilon(0)} \to \overline{B_\varepsilon(0)}$

$z^{[y]}(x) := y - z(x)$

$\rightsquigarrow$ $\|z^{[y]}(\tilde{x}) - z^{[y]}(x)\| \leq \frac{1}{2}\|\tilde{x} - x\|$

contraction!

Banach fixed-point theorem

$\Longrightarrow$ there is exactly one fixed point x^* : $z^{[y]}(x^*) = x^*$

Note: $z^{[y]}(x) = x \Leftrightarrow y - z(x) = x \Leftrightarrow y - \big(h(x) - x\big) = x$

$$\Leftrightarrow y = h(x)$$

Result: For each $y \in \overline{B_{\frac{\varepsilon}{2}}(0)}$, there is exactly one $x^* \in \overline{B_\varepsilon(0)}$ with

$$h(x^*) = y.$$

$\Rightarrow$ h is invertible as a map $h : \tilde{U} \to \hat{V}$, where $\tilde{U} = h^{-1}[\hat{V}]$ and $\hat{V} = B_{\frac{\varepsilon}{2}}(0)$

$\Rightarrow$ $h^{-1} : \hat{V} \to \tilde{U}$ is well-defined

To show:
(1) h^{-1} is continuous
(2) h^{-1} is differentiable
(3) $h^{-1} \in C^1(\hat{V}, \tilde{U})$

(1) $x, \tilde{x} \in U$: $\|\tilde{x} - x\| = \|h(\tilde{x}) - z(\tilde{x}) - h(x) + z(x)\|$ (using $z(x) := h(x) - x$)

$$\leq \|\underbrace{h(\tilde{x})}_{\tilde{y}} - \underbrace{h(x)}_{y}\| + \underbrace{\|z(\tilde{x}) - z(x)\|}_{\leq \frac{1}{2}\|\tilde{x} - x\|}$$

$$\Rightarrow \frac{1}{2}\|\tilde{x} - x\| \leq \|\tilde{y} - y\| \quad (x = h^{-1}(y))$$

$$\Rightarrow \|h^{-1}(y) - h^{-1}(\tilde{y})\| \leq 2\|\tilde{y} - y\| \Rightarrow h^{-1} \text{ is continuous}$$

(2) h is totally differentiable at x :

$$h(x) - h(\tilde{x}) = J_h(x)(x - \tilde{x}) + \phi(x - \tilde{x}) \quad \text{with} \quad \frac{\phi(x-\tilde{x})}{\|x - \tilde{x}\|} \xrightarrow{\tilde{x}\to x} 0$$

($h(x) = y$, $h(\tilde{x}) = \tilde{y}$, $x = h^{-1}(y)$, $\tilde{x} = h^{-1}(\tilde{y})$)

$$\Rightarrow J_h(x)^{-1}(y - \tilde{y}) + \underbrace{J_h(x)^{-1}(-1)\,\phi(x - \tilde{x})}_{\psi(y-\tilde{y})} = h^{-1}(y) - h^{-1}(\tilde{y})$$

($J_h(x)^{-1} = J_{h^{-1}}(y)$), $\psi(y - \tilde{y})$ satisfies: $\frac{\psi(y-\tilde{y})}{\|y - \tilde{y}\|} \xrightarrow{\tilde{y}\to y} 0$

(3) $J_{h^{-1}}(y) = J_h\big(h^{-1}(y)\big)^{-1}$ $\leftarrow$ composition of continuous functions

□