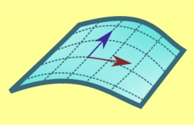
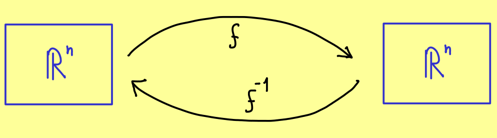
BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics



Multivariable Calculus – Part 21



differentiable in both directions
=
diffeomorphism

Recall: $C^k(\mathbb{R}^n, \mathbb{R}^m) = \left\{ f: \mathbb{R}^n \to \mathbb{R}^m \;\middle|\; \text{for all } f_j: \text{ all partial derivatives up to order } k \text{ exist and are continuous functions} \right\}$

Same for open sets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$: $C^k(U, V)$

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \begin{pmatrix} xy \\ 2 \end{pmatrix}$ $\Rightarrow$ $D^\alpha f(x,y) = \begin{pmatrix} y \\ 0 \end{pmatrix}$ for $\alpha = (1,0)$

$D^\alpha f(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $\alpha = (1,1)$

$$\Rightarrow f \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) = \bigcap_{k=1}^{\infty} C^k(\mathbb{R}^2, \mathbb{R}^2)$$

Definition: $U, V \subseteq \mathbb{R}^n$ open. A map $f: U \to V$ is called a C^k-diffeomorphism if:

(a) $f \in C^k(U, V)$

(b) f is bijective

(c) $f^{-1} \in C^k(V, U)$

Example: $f(x) = x^2$, $f: (0,\infty) \to (0,\infty)$ $\Rightarrow$ $f^{-1}(\tilde{x}) = \sqrt{\tilde{x}}$, $f^{-1}: (0,\infty) \to (0,\infty)$

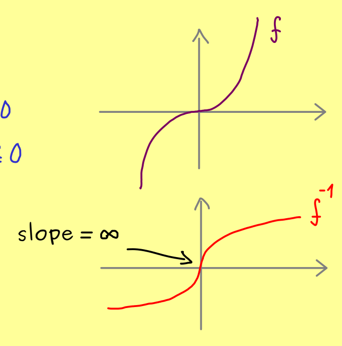
$\Rightarrow$ C^∞-diffeomorphism

Counterexample: $f(x) = x^3$, $f: \mathbb{R} \to \mathbb{R}$

$\Rightarrow f^{-1}: \mathbb{R} \to \mathbb{R}$, $\tilde{x} \mapsto \begin{cases} \sqrt[3]{\tilde{x}}, & \tilde{x} > 0 \\ -\sqrt[3]{-\tilde{x}} & \tilde{x} \leq 0 \end{cases}$

not differentiable at 0 !

$\Rightarrow$ f is not a C^1-diffeomorphism



Remember: $U, V \subseteq \mathbb{R}^n$ open, $f: U \to V$ C^1-diffeomorphism

$\Rightarrow f^{-1} \circ f = id_U$, $f \circ f^{-1} = id_V$

$\Rightarrow J_{f^{-1} \circ f}(x) = \mathbb{1}$, $J_{f \circ f^{-1}}(\tilde{x}) = \mathbb{1}$

chain rule ‖ $J_{f^{-1}}(f(x))\, J_f(x)$ chain rule ‖ $J_f(f^{-1}(\tilde{x}))\, J_{f^{-1}}(\tilde{x})$

$\Rightarrow J_f(x)$ invertible for all $x \in U$

$\Rightarrow \det(J_f(x)) \neq 0$ for all $x \in U$