The Bright Side of Mathematics

Multivariable Calculus – Part 19

$f: \mathbb{R}^n \longrightarrow \mathbb{R}\,, \quad x_0 \in \mathbb{R}^n\,, \quad \operatorname{grad} f(x_0) = 0$

$H_f(x_0)$ pos. definite $\Longrightarrow$ isolated local minimum at x_0

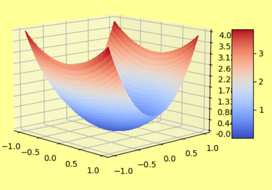
$H_f(x_0)$ neg. definite $\Longrightarrow$ isolated local maximum at x_0

Remember:

$H_f(x_0)$ pos. definite $\Longleftrightarrow$ all eigenvalues of $H_f(x_0)$ are > 0

$H_f(x_0)$ neg. definite $\Longleftrightarrow$ all eigenvalues of $H_f(x_0)$ are < 0

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}\,, \quad f(x) = x_1^2 + 3x_2^2$



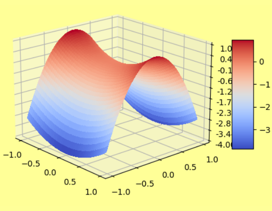
$$\operatorname{grad} f(x) = \begin{pmatrix} 2x_1 \\ 6x_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$H_f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$ ← diagonal matrix (eigenvalues on the diagonal)

$\Longrightarrow$ $H_f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ pos. definite

$\Longrightarrow$ f has an isolated local minimum at $(0,0)$

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}\,, \quad f(x) = x_1^2 - 4x_2^2$



$$\operatorname{grad} f(x) = \begin{pmatrix} 2x_1 \\ -8 \cdot x_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$H_f(x) = \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} \Longrightarrow H_f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$ indefinite $\Longrightarrow$ saddle point at $(0,0)$