BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Multivariable Calculus – Part 3

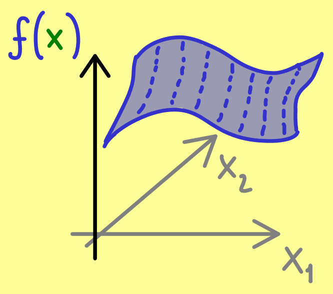
Definition: $f : \mathcal{D} \to \mathbb{R}^m$, $\mathcal{D} \subseteq \mathbb{R}^n$, is called continuous at $x \in \mathcal{D}$ if

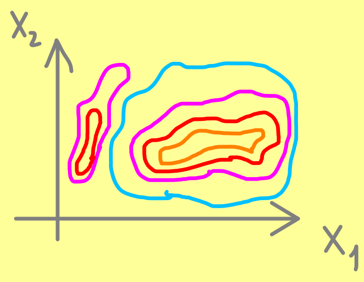
$$x^{(k)} \xrightarrow{k \to \infty} x \quad \Longrightarrow \quad f(x^{(k)}) \xrightarrow{k \to \infty} f(x)$$

$f : \mathcal{D} \to \mathbb{R}^m$ is called continuous

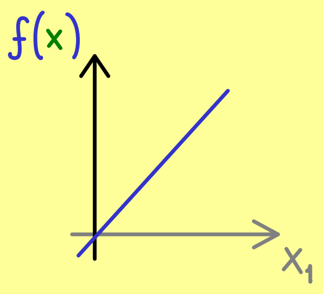
if f is continuous at all points $x \in \mathcal{D}$.

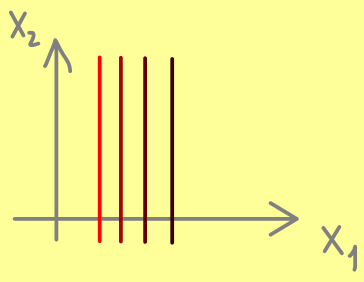
Examples: $f : \mathbb{R}^2 \to \mathbb{R}$





(a) $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1$





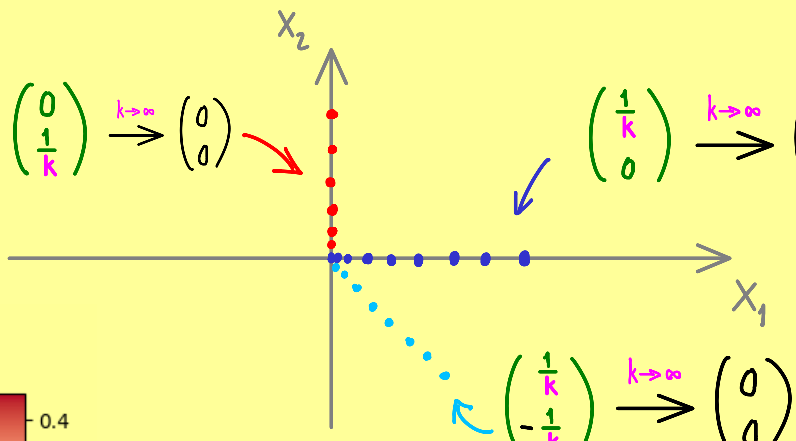
is continuous function!

For $x^{(k)} \xrightarrow{k \to \infty} x$, we have $f(x^{(k)}) = x_1^{(k)} \xrightarrow{k \to \infty} x_1 = f(x)$

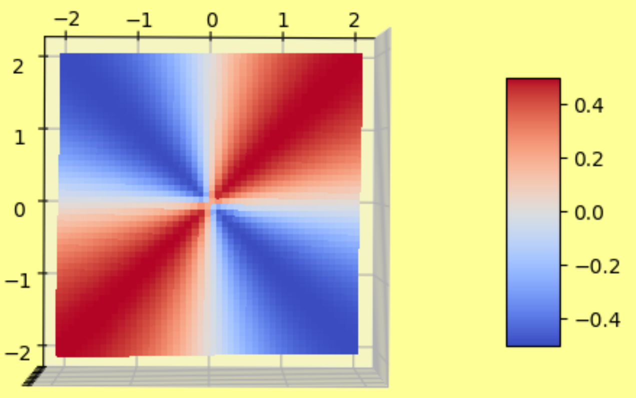
(b) $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2} & , \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & , \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$

$\begin{pmatrix} 0 \\ \frac{1}{k} \end{pmatrix} \xrightarrow{k \to \infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Leftarrow$ $f\left(\begin{pmatrix} 0 \\ \frac{1}{k} \end{pmatrix}\right) = 0 \xrightarrow{k \to \infty} 0$

$\begin{pmatrix} \frac{1}{k} \\ 0 \end{pmatrix} \xrightarrow{k \to \infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow f\left(\begin{pmatrix} \frac{1}{k} \\ 0 \end{pmatrix}\right) = 0 \xrightarrow{k \to \infty} 0$

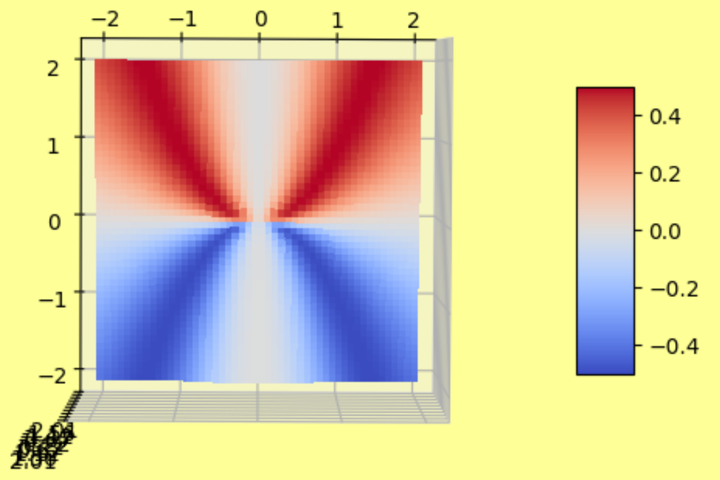


$\begin{pmatrix} \frac{1}{k} \\ -\frac{1}{k} \end{pmatrix} \xrightarrow{k \to \infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$$\Longrightarrow f\left(\begin{pmatrix} \frac{1}{k} \\ -\frac{1}{k} \end{pmatrix}\right) = \frac{\frac{1}{k} \cdot \left(-\frac{1}{k}\right)}{\frac{1}{k^2} + \frac{1}{k^2}} = -\frac{1}{2} \frac{\cancel{\frac{1}{k^2}}}{\cancel{\frac{1}{k^2}}} = -\frac{1}{2} \xrightarrow{k \to \infty} -\frac{1}{2}$$

(c) $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{cases} \dfrac{x_1^2 x_2}{x_1^4 + x_2^2} & , \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & , \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$



$$\begin{pmatrix} \frac{1}{k} \\ \frac{1}{k^2} \end{pmatrix} \xrightarrow{k \to \infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow f\left(\begin{pmatrix} \frac{1}{k} \\ \frac{1}{k^2} \end{pmatrix}\right) = \frac{\frac{1}{k^2} \left(\frac{1}{k^2}\right)}{\frac{1}{k^4} + \frac{1}{k^4}} = \frac{1}{2} \xrightarrow{k \to \infty} \frac{1}{2}$$