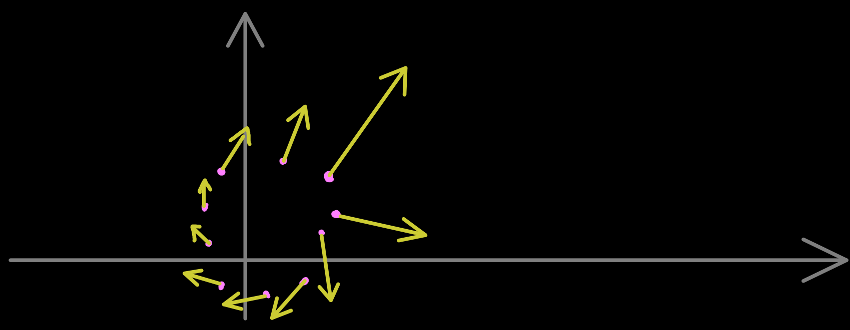
Multivariable Calculus – Part 14

Vector field: $v: \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuously differentiable

$v: \mathbb{R}^2 \rightarrow \mathbb{R}^2$



Question: Is there a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with $\operatorname{grad} f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

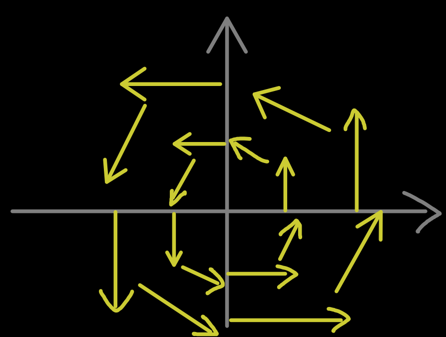
satisfying $v(x) = \operatorname{grad} f(x)$ for all $x \in \mathbb{R}^n$. $\left(f \in C^2(\mathbb{R}^n) \right)$

potential function for v

Necessary condition:

$$\boxed{\begin{matrix} \dfrac{\partial}{\partial x_j} v_i(x) \\ \| \\ \dfrac{\partial}{\partial x_i} v_j(x) \end{matrix}} \begin{matrix} = \dfrac{\partial}{\partial x_j}\left(\operatorname{grad} f(x)\right)_i = \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial}{\partial x_i} f(x)\right) \\ \| \text{ Schwarz's theorem} \\ = \dfrac{\partial}{\partial x_i}\left(\operatorname{grad} f(x)\right)_j = \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial}{\partial x_j} f(x)\right) \end{matrix}$$

Example: $v: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$



$$\frac{\partial v_1}{\partial x_2} = -1$$

$$\#$$

$$\frac{\partial v_2}{\partial x_1} = 1 \quad \Longrightarrow \quad \text{there is no potential function } f$$