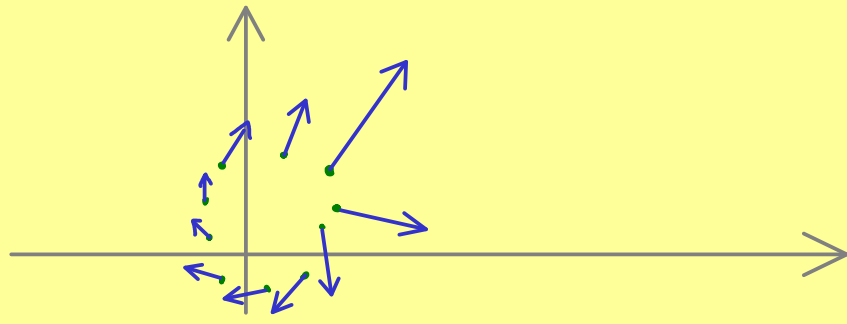The Bright Side of
Mathematics

Multivariable Calculus – Part 14

<u>Vector field:</u> $V: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuously differentiable

$V: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

<u>Question:</u>   Is there a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\text{grad} f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

satisfying $V(x) = \text{grad} f(x)$ for all $x \in \mathbb{R}^n$. $\left(f \in C^2(\mathbb{R}^n) \right)$

<u>potential function for V</u>
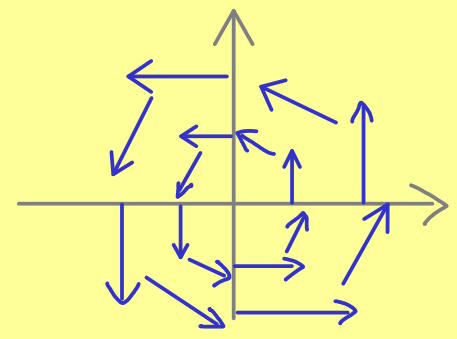
<u>Necessary condition:</u>

$$\frac{\partial}{\partial x_j} V_i(x) = \frac{\partial}{\partial x_j} \left(\text{grad} f(x) \right)_i = \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_i} f(x) \right)$$

$$\|$$

$\|$ Schwarz's theorem

$$\frac{\partial}{\partial x_i} V_j(x) = \frac{\partial}{\partial x_i} \left(\text{grad} f(x) \right)_j = \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j} f(x) \right)$$

<u>Example:</u> $V: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

$$\frac{\partial V_1}{\partial x_2} = -1$$

$$\frac{\partial V_2}{\partial x_1} = 1 \qquad \Longrightarrow \quad \text{there is no potential function } f$$