The Bright Side of Mathematics

Multivariable Calculus – Part 11

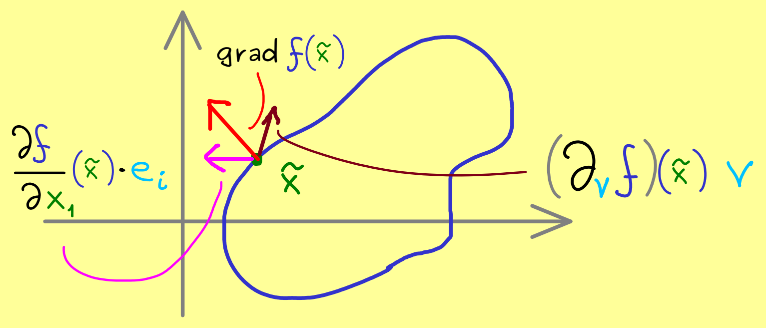
$f: \mathbb{R}^n \to \mathbb{R}$ (totally) differentiable

Directional derivative: $(\partial_v f)(\tilde{x}) = \langle \operatorname{grad} f(\tilde{x}), v \rangle$

For $v = e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i-th component (canonical unit vector)

$$(\partial_{e_i} f)(\tilde{x}) = \langle \operatorname{grad} f(\tilde{x}), e_i \rangle = \frac{\partial f}{\partial x_i}(\tilde{x})$$

Visualisation: $f: \mathbb{R}^2 \to \mathbb{R}$



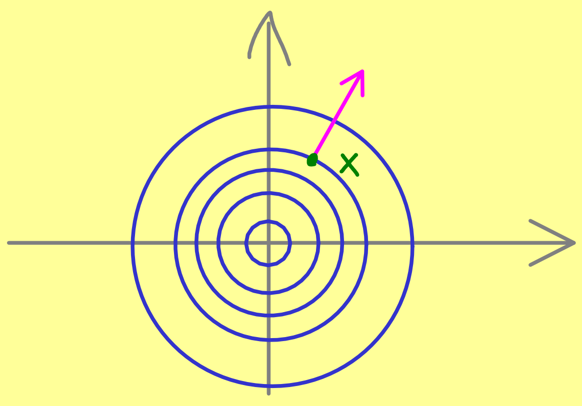
Question: For which $v \in \mathbb{R}^n$ is $(\partial_v f)(\tilde{x})$ maximal? $(\|v\| = 1)$

Answer: $(\partial_v f)(\tilde{x}) = \langle \operatorname{grad} f(\tilde{x}), v \rangle = \|\operatorname{grad} f(\tilde{x})\| \cdot \underbrace{\|v\|}_{=1} \cdot \underbrace{\cos(\alpha)}_{\in [-1,1]}$, $\quad \alpha = \measuredangle(\operatorname{grad} f(\tilde{x}), v)$

maximal $\iff$ v shows into the direction of $\operatorname{grad} f(\tilde{x})$

$$\boxed{\text{gradient} = \text{direction with the fastest increase}}$$

Example: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_1^2 + x_2^2$



$$\operatorname{grad} f(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$