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ON STEADY

The Bright Side of Mathematics

Multivariable Calculus – Part 7

Total differentiation is linear: $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^n \to \mathbb{R}^m$

(totally) differentiable at $\tilde{x} \in \mathbb{R}^n$. Then:

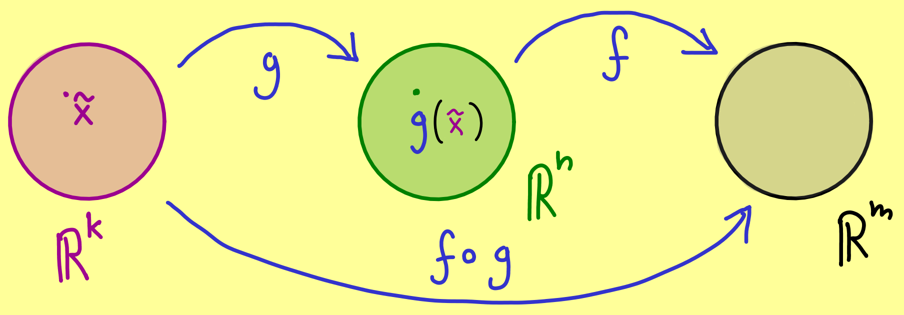
(a) $f+g: \mathbb{R}^n \to \mathbb{R}^m$ is (totally) differentiable at $\tilde{x} \in \mathbb{R}^n$ and

$$d(f+g)_{\tilde{x}} = df_{\tilde{x}} + dg_{\tilde{x}} \qquad \big(J_{f+g}(\tilde{x}) = J_f(\tilde{x}) + J_g(\tilde{x}) \big)$$

(b) For all $\lambda \in \mathbb{R}$, $\lambda \cdot f: \mathbb{R}^n \to \mathbb{R}^m$ is (totally) differentiable at $\tilde{x} \in \mathbb{R}^n$ and

$$d(\lambda \cdot f)_{\tilde{x}} = \lambda \cdot df_{\tilde{x}} \qquad \big(J_{\lambda f}(\tilde{x}) = \lambda \cdot J_f(\tilde{x}) \big)$$

Chain rule:



$g: \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$

$(f \circ g)'(\tilde{x}) = f'(g(\tilde{x})) \cdot g'(\tilde{x})$

Now: $g: \mathbb{R}^k \to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^m$

(totally) differentiable at $\tilde{x}$ (totally) differentiable at $g(\tilde{x})$

Then: $$d(f \circ g)_{\tilde{x}} = df_{g(\tilde{x})} \circ dg_{\tilde{x}} \qquad \big(J_{f \circ g}(\tilde{x}) = J_f(g(\tilde{x})) \cdot J_g(\tilde{x}) \big)$$

Sketch of the proof: We have:

$$g(\tilde{x}+h) = g(\tilde{x}) + dg_{\tilde{x}}(h) + \phi_g(h) \quad \text{for all } h \in \mathbb{R}^k$$

$$f(g(\tilde{x})+h) = f(g(\tilde{x})) + df_{g(\tilde{x})}(h) + \phi_f(h) \quad \text{for all } h \in \mathbb{R}^n$$

$$\Rightarrow (f \circ g)(\tilde{x}+h) = f(g(\tilde{x}+h)) = f\big(g(\tilde{x}) + \underbrace{dg_{\tilde{x}}(h) + \phi_g(h)}_{h_1}\big)$$

$$= f\big(g(\tilde{x}) + h_1\big)$$

$$= f(g(\tilde{x})) + df_{g(\tilde{x})}(h_1) + \phi_f(h_1)$$

$$= f(g(\tilde{x})) + df_{g(\tilde{x})}\big(dg_{\tilde{x}}(h) + \phi_g(h)\big) + \phi_f(h_1)$$

$$= f(g(\tilde{x})) + df_{g(\tilde{x})}\big(dg_{\tilde{x}}(h)\big) + \underbrace{df_{g(\tilde{x})}\big(\phi_g(h)\big) + \phi_f(h_1)}_{\phi_{f \circ g}}$$