BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

Multivariable Calculus – Part 6

Definition: $f: U \to \mathbb{R}^m$ is called (totally) differentiable at $\tilde{x} \in U$ if there is a linear map $\ell: \mathbb{R}^n \to \mathbb{R}^m$ and a map $\phi: \mathbb{R}^n \to \mathbb{R}^m$ with:

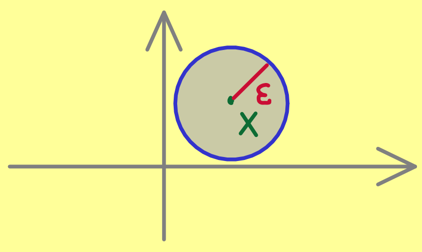
$U \subseteq \mathbb{R}^n$ open!

$$f(\tilde{x}+h) = f(\tilde{x}) + \ell(h) + \phi(h) \quad \text{for all } h \in \mathbb{R}^n \quad \left(\text{with } \tilde{x}+h \in U\right)$$

$$\text{with} \quad \frac{\phi(h)}{\|h\|} \xrightarrow{h \to 0} 0 \quad \left(\text{means } h_k \xrightarrow{k\to\infty} 0 \implies \frac{\phi(h_k)}{\|h_k\|} \xrightarrow{k\to\infty} 0\right)$$

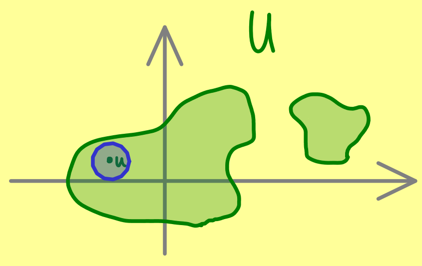
Definition: $B_\varepsilon(x) := \left\{ a \in \mathbb{R}^n \mid d_{\text{Euclid}}(a, x) < \varepsilon \right\}$

ε-neighbourhood of x



A set $U \subseteq \mathbb{R}^n$ is called open (in $\mathbb{R}^n$) if

$$\forall u \in U \ \exists \varepsilon > 0 : \quad U \supseteq B_\varepsilon(u)$$



Proposition: $f: U \to \mathbb{R}^m$ with $U \subseteq \mathbb{R}^n$ open. $f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$

(a) f is (totally) differentiable at $\tilde{x} \in U$

$$\implies \begin{cases} f \text{ is continuous at } \tilde{x} \in U \\ \dfrac{\partial f}{\partial x_i}(\tilde{x}) \text{ exists for all } i \in \{1, \dots, n\} \text{ where } \dfrac{\partial f}{\partial x_i}(\tilde{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(\tilde{x}) \\ \vdots \\ \frac{\partial f_m}{\partial x_i}(\tilde{x}) \end{pmatrix} \\ J_f(\tilde{x}) = \left(\dfrac{\partial f}{\partial x_1}(\tilde{x}) \ \ \dfrac{\partial f}{\partial x_2}(\tilde{x}) \cdots \dfrac{\partial f}{\partial x_n}(\tilde{x}) \right) \end{cases}$$

(b) If $\dfrac{\partial f}{\partial x_i}(\tilde{x})$ exists for all $i \in \{1, \dots, n\}$ and for all $\tilde{x} \in U$ and $\dfrac{\partial f}{\partial x_i}: U \to \mathbb{R}^m$ is continuous for all $i \in \{1, \dots, n\}$,

then: f is (totally) differentiable at all $\tilde{x} \in U$.

Example: $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x_1, x_2) = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^3 \\ x_2 \end{pmatrix}$

$$\frac{\partial f}{\partial x_1}(x) = \begin{pmatrix} 2x_1 \\ 3x_1^2 \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial x_2}(x) = \begin{pmatrix} 2x_2 \\ 0 \\ 1 \end{pmatrix} \quad \text{continuous functions!}$$

$$\implies f \text{ is (totally) differentiable with } J_f(x) = \begin{pmatrix} 2x_1 & 2x_2 \\ 3x_1^2 & 0 \\ 0 & 1 \end{pmatrix}$$