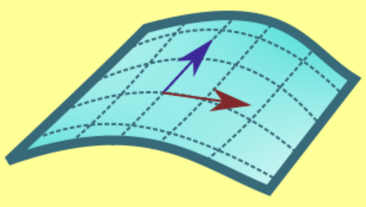
The Bright Side of Mathematics



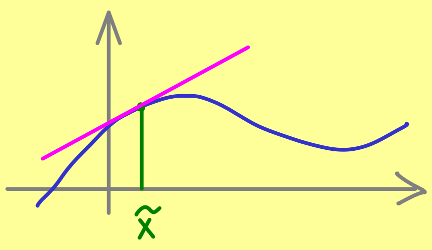
Multivariable Calculus – Part 5

Linear approximation: (Real Analysis Part 34/35)

$f: \mathbb{R} \to \mathbb{R}$ is called differentiable at $\tilde{x} \in \mathbb{R}$ if

there is a number $b \in \mathbb{R}$

and a function $r: \mathbb{R} \to \mathbb{R}$ with:



$$f(\tilde{x}+h) = f(\tilde{x}) + \underbrace{b}_{= f'(\tilde{x})} \cdot h + r(h) \cdot h \quad \text{for all } h \in \mathbb{R}$$

$$\text{with } r(h) \xrightarrow{h \to 0} 0$$

Linear approximation in higher dimensions: linear map $\ell: \mathbb{R}^n \to \mathbb{R}^m$

instead of $h \mapsto b \cdot h$

Definition: $f: \mathbb{R}^n \to \mathbb{R}^m$ is called (totally) differentiable at $\tilde{x} \in \mathbb{R}^n$ if

there is a linear map $\ell: \mathbb{R}^n \to \mathbb{R}^m$ and a map $\phi: \mathbb{R}^n \to \mathbb{R}^m$ with:

$$f(\tilde{x}+h) = f(\tilde{x}) + \ell(h) + \phi(h) \quad \text{for all } h \in \mathbb{R}^n$$

$$\text{with } \frac{\phi(h)}{\|h\|} \xrightarrow{h \to 0} 0$$

Euclidean norm

$$\| h \| = d_{\text{Euclid}}(h, 0) = \sqrt{h_1^2 + h_2^2 + \cdots + h_n^2}$$

Instead of $\ell(h)$, one often writes: $\underbrace{df_{\tilde{x}}}(h)$, $\underbrace{Df_{\tilde{x}}}(h)$ or $\underbrace{J_f(\tilde{x})}\, h$

(total) derivative of f at $\tilde{x}$ — Jacobian matrix of f at $\tilde{x}$

For $n=1$, $m=1$: $J_f(\tilde{x}) = \left(f'(\tilde{x}) \right)$ 1x1-matrix

$$df_{\tilde{x}}(h) = f'(\tilde{x}) \cdot h$$

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ totally differentiable at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$?

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) = \begin{pmatrix} h_2 \\ h_1 \end{pmatrix} = \underbrace{f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}_{0} + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{J_f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \underbrace{\phi(h)}_{=0}$$