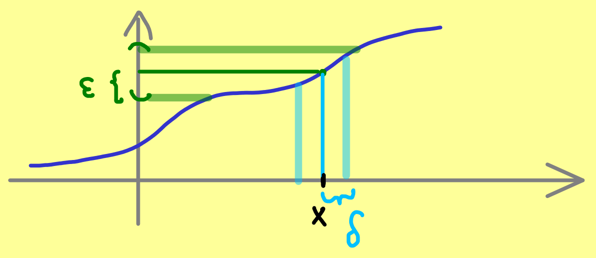
The Bright Side of Mathematics

Multivariable Calculus – Part 2

Continuity: $f: \mathbb{R} \to \mathbb{R}$



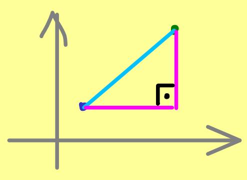
ε-δ-definition ← (measure distance on x-axis and on y-axis)

alternative definition with sequences:

$$x_k \xrightarrow{k\to\infty} x \implies f(x_k) \xrightarrow{k\to\infty} f(x)$$

Euclidean distance: $f: \mathbb{R}^n \to \mathbb{R}^m$ ← how to measure distances?

in $\mathbb{R}^2$



for $x, \tilde{x} \in \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix}$

$$d_{\text{Euclid}}(x, \tilde{x}) := \sqrt{(x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 + \cdots + (x_n - \tilde{x}_n)^2}$$

Definition: A sequence $(x^{(k)})_{k\in\mathbb{N}}$, $x^{(k)} \in \mathbb{R}^n$, is called convergent to $x \in \mathbb{R}^n$

if $\forall \varepsilon > 0 \quad \exists K \in \mathbb{N} \quad \forall k \geq K : \quad d_{\text{Euclid}}(x^{(k)}, x) < \varepsilon$

In this case, we write: $\lim\limits_{k\to\infty} x^{(k)} = x$ or $x^{(k)} \xrightarrow{k\to\infty} x$

Fact:

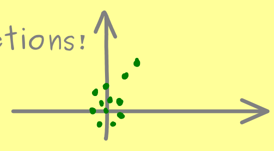
$$\lim_{k\to\infty} \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \iff \lim_{k\to\infty} x_j^{(k)} = x_j \quad \text{for all } j \in \{1, \dots, n\}$$

one needs to prove it!

Definition: $f: \mathbb{R}^n \to \mathbb{R}^m$ is called continuous at $x \in \mathbb{R}^n$ if

for all sequences $(x^{(k)})_{k\in\mathbb{N}}$ we have:

a lot of possible directions!



$$x^{(k)} \xrightarrow{k\to\infty} x \implies f(x^{(k)}) \xrightarrow{k\to\infty} f(x)$$