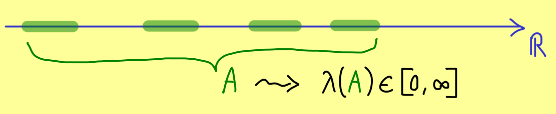
BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Multidimensional Integration - Part 2



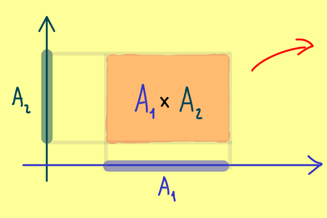
collection of subsets:

$\mathcal{L}(\mathbb{R})$ ← of Lebesgue-measurable sets

σ-algebra

Now go to $\mathbb{R}^2$: $\mathbb{R} \times \mathbb{R}$ (construction of product measure)

$A_1 \in \mathcal{L}(\mathbb{R})$
$A_2 \in \mathcal{L}(\mathbb{R})$

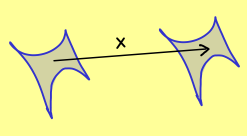


rectangle in $\mathbb{R}^2$

area should be: $\lambda(A_1) \cdot \lambda(A_2)$

We get:

- product σ-algebra $\xrightarrow{\text{extension/completion}}$ $\mathcal{L}(\mathbb{R}^2)$ (Lebesgue-measurable subsets of $\mathbb{R}^2$)
- product measure $\lambda^{(2)}: \mathcal{L}(\mathbb{R}^2) \longrightarrow [0,\infty]$ Lebesgue measure on $\mathbb{R}^2$
- $\lambda^{(2)}(A_1 \times A_2) = \lambda(A_1) \cdot \lambda(A_2)$ for $A_1 \in \mathcal{L}(\mathbb{R}), A_2 \in \mathcal{L}(\mathbb{R})$
- properties like for the one-dimensional Lebesgue measure:
  - $\lambda^{(2)}(\emptyset) = 0$
  - $\lambda^{(2)}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda^{(2)}(A_j)$ for $A_j \in \mathcal{L}(\mathbb{R}^2)$, $A_j \cap A_i = \emptyset$ for $i \neq j$
  - $\mathcal{L}(\mathbb{R}^2)$ is larger than the Borel σ-algebra.
  - If $A \in \mathcal{L}(\mathbb{R}^2)$ with $\lambda^{(2)}(A) = 0$, (A is called null set) then each $B \subseteq A$ satisfies $B \in \mathcal{L}(\mathbb{R}^2)$.
  - $\lambda^{(2)}([0,1) \times [0,1)) = 1$ (unit square as area 1)
  - $\lambda^{(2)}(x + A) = \lambda^{(2)}(A)$ for all $x \in \mathbb{R}^2$, $A \in \mathcal{L}(\mathbb{R}^2)$ (translation-invariant)



We call $\lambda^{(2)}$ the two-dimensional Lebesgue measure!

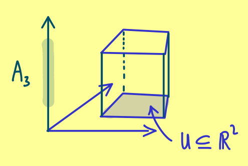
→ the corresponding Lebesgue integral: $\int_A f \, d\lambda^{(2)}$

the two-dimensional Lebesgue integral

Other notations:

$$\int_A f \, d\lambda^{(2)} = \int_A f(x) \, d\lambda^{(2)}(x)$$
$$= \int_A f(x_1, x_2) \, d\lambda^{(2)}(x_1, x_2) = \int_A f(x_1, x_2) \, d(x_1, x_2)$$
$$= \int_A f(x) \, d^2x$$

Do it again!



volume in $\mathbb{R}^3$: $\lambda^{(2)}(U) \cdot \lambda(A_3)$ ⇝ $\lambda^{(3)}$ as a product measure

Result: n-dimensional Lebesgue measure on $\mathbb{R}^n$: $\lambda^{(n)}: \mathcal{L}(\mathbb{R}^n) \longrightarrow [0,\infty]$

properties:

- $\lambda^{(n)}(\emptyset) = 0$
- $\lambda^{(n)}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda^{(n)}(A_j)$ for $A_j \in \mathcal{L}(\mathbb{R}^n)$, $A_j \cap A_i = \emptyset$ for $i \neq j$
- $\mathcal{L}(\mathbb{R}^n)$ is larger than the Borel σ-algebra.
- If $A \in \mathcal{L}(\mathbb{R}^n)$ with $\lambda^{(n)}(A) = 0$, (A is called null set) then each $B \subseteq A$ satisfies $B \in \mathcal{L}(\mathbb{R}^n)$.
- $\lambda^{(n)}([0,1) \times [0,1) \times \cdots \times [0,1)) = 1$
- $\lambda^{(n)}(x + A) = \lambda^{(n)}(A)$ for all $x \in \mathbb{R}^n$, $A \in \mathcal{L}(\mathbb{R}^n)$ (translation-invariant)

n-dimensional Lebesgue integral: $\int_A f \, d\lambda^{(n)} = \int_A f(x) \, d^n x$