

## Multidimensional Integration - Part 4

Fubini's theorem (Fubini-Tonelli theorem): Let  $f$  be measurable with

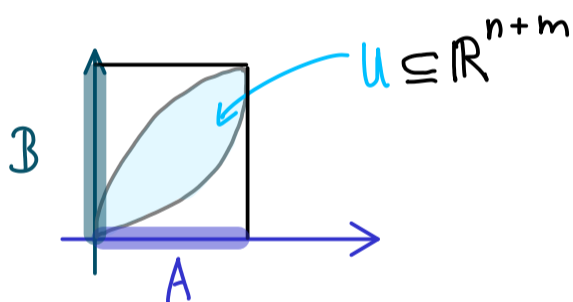
$$\text{either } f: A \times B \rightarrow [0, \infty] \quad \left( A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m \right)$$

$$\text{or } f: A \times B \rightarrow \mathbb{R} \text{ with } \int_{A \times B} |f| d\lambda^{(n+m)} < \infty.$$

Then:

$$\int_{A \times B} f d\lambda^{(n+m)} = \int_A \left( \int_B f(x, y) d^m y \right) d^n x = \int_B \left( \int_A f(x, y) d^n x \right) d^m y$$

Problem:



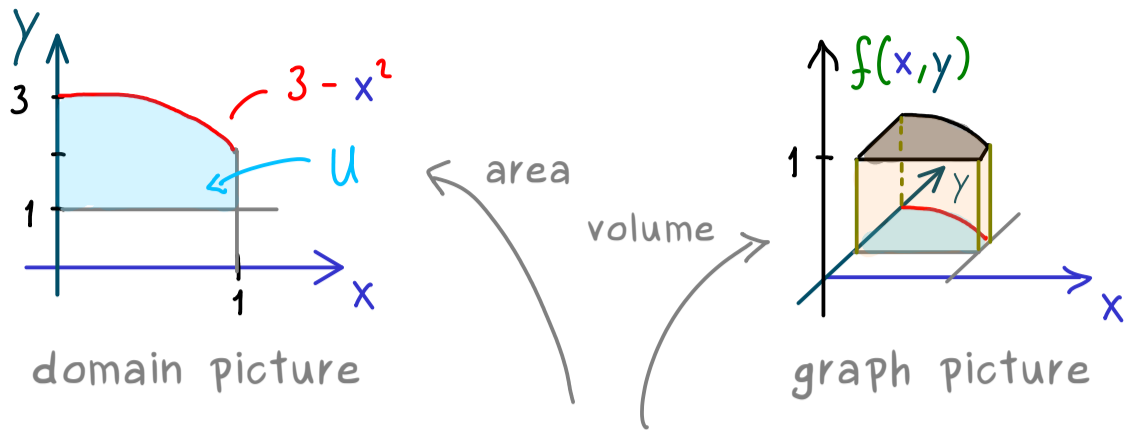
$$f: U \rightarrow \mathbb{R}$$

$$\hookrightarrow \tilde{f}: A \times B \rightarrow \mathbb{R}$$

$$(x, y) \mapsto \begin{cases} f(x, y) & \text{if } (x, y) \in U \\ 0 & \text{if } (x, y) \notin U \end{cases}$$

$$\int_U f d\lambda^{(n+m)} = \int_{A \times B} \tilde{f} d\lambda^{(n+m)} \stackrel{\text{Fubini}}{=} \int_A \left( \int_B \tilde{f}(x, y) d^m y \right) d^n x$$

Example:



$$\int 1 d(x,y)$$

$$\int_{[0,1] \times [1,3]} \tilde{f}(x,y) d(x,y) \quad \text{with } \tilde{f}(x,y) := \begin{cases} 1, & x \in [0,1], y \in [1, 3-x^2] \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} & \stackrel{\text{Fubini}}{=} \int_0^1 \left( \int_1^{3-x^2} \tilde{f}(x,y) dy \right) dx = \int_0^1 \left( \int_1^{3-x^2} 1 dy \right) dx \\ & = \int_0^1 (3-x^2-1) dx = \int_0^1 (2-x^2) dx = \frac{5}{3} \end{aligned}$$