Multidimensional Integration - Part 4

Fubini's theorem (Fubini-Tonelli theorem): Let f be measurable with

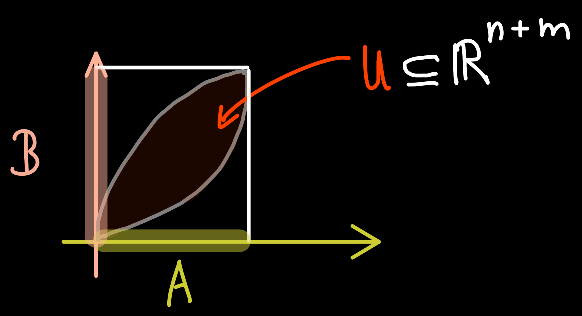
either $f: A \times B \longrightarrow [0,\infty]$ $\qquad (A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m)$

or $f: A \times B \longrightarrow \mathbb{R}$ with $\int_{A\times B} |f| \, d\lambda^{(n+m)} < \infty$.

Then:

$$\int_{A\times B} f \, d\lambda^{(n+m)} = \int_A \left(\int_B f(x,y) \, d^m y \right) d^n x = \int_B \left(\int_A f(x,y) \, d^n x \right) d^m y$$

Problem:



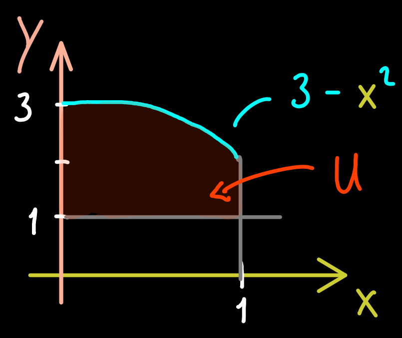
$f: U \longrightarrow \mathbb{R}$

$\hookrightarrow \tilde{f}: A \times B \longrightarrow \mathbb{R}$

$$(x,y) \mapsto \begin{cases} f(x,y) & \text{if } (x,y) \in U \\ 0 & \text{if } (x,y) \notin U \end{cases}$$

$$\int_U f \, d\lambda^{(n+m)} = \int_{A\times B} \tilde{f} \, d\lambda^{(n+m)} \overset{\text{Fubini}}{=} \int_A \left(\int_B \tilde{f}(x,y) \, d^m y \right) d^n x$$

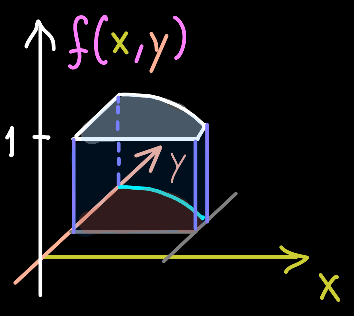
Example:



domain picture

area

volume



graph picture

$$\int_U 1 \, d(x,y)$$

$$= \int_{[0,1]\times[1,3]} \tilde{f}(x,y) \, d(x,y) \quad \text{with} \quad \tilde{f}(x,y) := \begin{cases} 1 \,, & x \in [0,1], y \in [1, 3-x^2] \\ 0 & \text{else} \end{cases}$$

Fubini

$$= \int_0^1 \left(\int_1^3 \tilde{f}(x,y) \, dy \right) dx = \int_0^1 \left(\int_1^{3-x^2} 1 \, dy \right) dx$$

$$= \int_0^1 \left(3 - x^2 - 1 \right) dx = \int_0^1 (2 - x^2) \, dx = \frac{5}{3}$$