The Bright Side of Mathematics

The following pages cover the whole Multidimensional Integration course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

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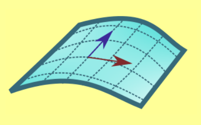
The Bright Side of
Mathematics



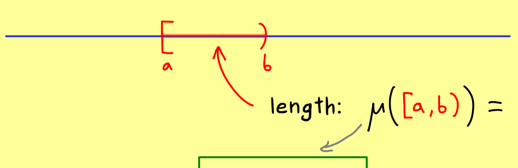
Multidimensional Integration - Part 1

Measure Theory → Multidimensional Integration ← Multivariable Calculus

Multidimensional Integration → Manifolds

- Definitions for integrals
- Change-of-variables formula
- Fubini's theorem
  ↳ (Fubini-Tonelli theorem)
- Explicit calculations

Lebesgue measure on $\mathbb{R}$:



$a \leq b$

length: $\mu([a,b)) = b - a$

$\boxed{\text{pre-measure}}$

Carathéodory's Extension Theorem

$\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty]$ $\boxed{\text{outer measure}}$

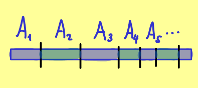
$\mathcal{L}(\mathbb{R}) = \mathcal{A}_\varphi$ σ-algebra of Lebesgue-measurable sets

$\lambda : \mathcal{L}(\mathbb{R}) \longrightarrow [0,\infty]$ Lebesgue measure on $\mathbb{R}$

$\boxed{\text{measure}}$

Properties:

- $\lambda(\emptyset) = 0$
- $\lambda\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda(A_j)$ for $A_j \in \mathcal{L}(\mathbb{R})$
  $A_j \cap A_i = \emptyset$ for $i \neq j$



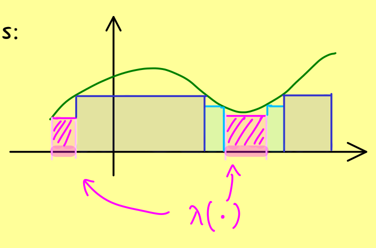
- $\mathcal{L}(\mathbb{R})$ is larger than the Borel σ-algebra.
- If $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) = 0$, (A is called null set)
  then each $B \subseteq A$ satisfies $B \in \mathcal{L}(\mathbb{R})$.
- $\lambda([a,b)) = b - a$, $b \geq a$
- $\lambda(x + A) = \lambda(A)$ for all $x \in \mathbb{R}$, $A \in \mathcal{L}(\mathbb{R})$

(translation-invariant)

Definition (Lebesgue integral):

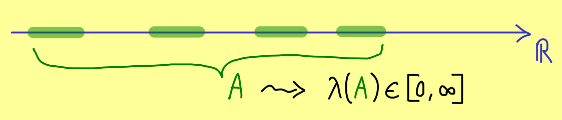
$$\int_A f \, d\lambda = \int_A f(x) \, d\lambda(x) = \int_A f(x) \, dx$$

defined by approximation with simple functions:



Multidimensional Integration - Part 2

collection of subsets:



$A \rightsquigarrow \lambda(A) \in [0,\infty]$ $\mathbb{R}$

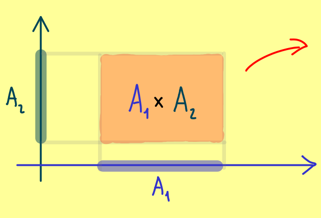
Lebsgue measure

$\mathcal{L}(\mathbb{R})$

σ- algebra of Lebesgue-measurable sets

Now go to $\mathbb{R}^2$: $\mathbb{R} \times \mathbb{R}$ (construction of product measure)

$A_1 \in \mathcal{L}(\mathbb{R})$
$A_2 \in \mathcal{L}(\mathbb{R})$

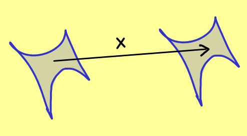


rectangle in $\mathbb{R}^2$

area should be: $\lambda(A_1) \cdot \lambda(A_2)$

We get:

- product σ- algebra $\xrightarrow{\text{extension/ completion}}$ $\mathcal{L}(\mathbb{R}^2)$ (Lebesgue-measurable subsets of $\mathbb{R}^2$)
- product measure $\lambda^{(2)}: \mathcal{L}(\mathbb{R}^2) \longrightarrow [0,\infty]$ Lebesgue measure on $\mathbb{R}^2$
- $\lambda^{(2)}(A_1 \times A_2) = \lambda(A_1) \cdot \lambda(A_2)$ for $A_1 \in \mathcal{L}(\mathbb{R})$, $A_2 \in \mathcal{L}(\mathbb{R})$
- properties like for the one-dimensional Lebesgue measure:
  - $\lambda^{(2)}(\emptyset) = 0$
  - $\lambda^{(2)}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda^{(2)}(A_j)$ for $A_j \in \mathcal{L}(\mathbb{R}^2)$, $A_j \cap A_i = \emptyset$ for $i \neq j$
  - $\mathcal{L}(\mathbb{R}^2)$ is larger than the Borel σ-algebra.
  - If $A \in \mathcal{L}(\mathbb{R}^2)$ with $\lambda^{(2)}(A) = 0$, (A is called null set) then each $B \subseteq A$ satisfies $B \in \mathcal{L}(\mathbb{R}^2)$.
  - $\lambda^{(2)}\big([0,1) \times [0,1)\big) = 1$ (unit square as area 1)
  - $\lambda^{(2)}(x + A) = \lambda^{(2)}(A)$ for all $x \in \mathbb{R}^2$, $A \in \mathcal{L}(\mathbb{R}^2)$



(translation-invariant)

We call $\lambda^{(2)}$ the two-dimensional Lebesgue measure!

$\hookrightarrow$ the corresponding Lebesgue integral: $\int_A f \, d\lambda^{(2)}$

the two-dimensional Lebesgue integral

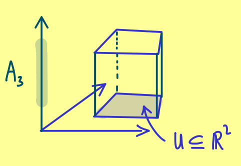
Other notations:

$$\int_A f \, d\lambda^{(2)} = \int_A f(x) \, d\lambda^{(2)}(x)$$

$$= \int_A f(x_1, x_2) \, d\lambda^{(2)}(x_1, x_2) = \int_A f(x_1, x_2) \, d(x_1, x_2)$$

$$= \int_A f(x) \, d^2x$$

Do it again!



volume in $\mathbb{R}^3$: $\lambda^{(2)}(U) \cdot \lambda(A_3)$ $\rightsquigarrow$ $\lambda^{(3)}$ as a product measure

Result: n-dimensional Lebesgue measure on $\mathbb{R}^n$: $\lambda^{(n)}: \mathcal{L}(\mathbb{R}^n) \longrightarrow [0,\infty]$

properties:

- $\lambda^{(n)}(\emptyset) = 0$
- $\lambda^{(n)}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda^{(n)}(A_j)$ for $A_j \in \mathcal{L}(\mathbb{R}^n)$, $A_j \cap A_i = \emptyset$ for $i \neq j$
- $\mathcal{L}(\mathbb{R}^n)$ is larger than the Borel σ-algebra.
- If $A \in \mathcal{L}(\mathbb{R}^n)$ with $\lambda^{(n)}(A) = 0$, (A is called null set) then each $B \subseteq A$ satisfies $B \in \mathcal{L}(\mathbb{R}^n)$.
- $\lambda^{(n)}\big([0,1) \times [0,1) \times \cdots \times [0,1)\big) = 1$
- $\lambda^{(n)}(x + A) = \lambda^{(n)}(A)$ for all $x \in \mathbb{R}^n$, $A \in \mathcal{L}(\mathbb{R}^n)$

(translation-invariant)

n-dimensional Lebesgue integral: $\int_A f \, d\lambda^{(n)} = \int_A f(x) \, d^n x$

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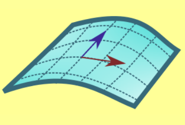
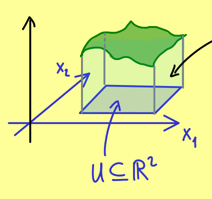


Multidimensional Integration - Part 3

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



two-dimensional Lebesgue integral

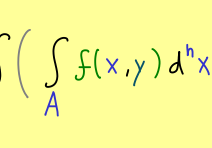
$\int_U f(x_1, x_2)\, d(x_1, x_2)$

Example:

$$\underbrace{\int_{[0,1]\times[0,2]} x_1^2 \cdot x_2 \, d(x_1,x_2)}_{\text{two-dimensional Lebesgue integral}} \overset{\text{Fubini's theorem}}{=} \underbrace{\int_0^1 \Big(\underbrace{\int_0^2 x_1^2 \cdot x_2 \, dx_2}_{\text{one-dimensional Lebesgue integral}} \Big) dx_1}_{\text{one-dimensional Lebesgue integral}}$$

Fubini's theorem (Fubini-Tonelli theorem)

Let $\lambda^{(n)}$ be the n-dimensional Lebesgue measure on $\mathbb{R}^n$ and $\lambda^{(m)}$ be the m-dimensional Lebesgue measure on $\mathbb{R}^m$.



Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$, and f be a measurable function with

either $f: A \times B \longrightarrow [0,\infty]$

or $f: A \times B \longrightarrow \mathbb{R}$ with $\int_{A\times B} |f| \, d\lambda^{(n+m)} < \infty$.

Then:

$$\int_{A\times B} f \, d\lambda^{(n+m)} = \int_A \Big(\int_B f(x,y)\, d^m y \Big) d^n x = \int_B \Big(\int_A f(x,y)\, d^n x \Big) d^m y$$

Multidimensional Integration - Part 4

Fubini's theorem (Fubini-Tonelli theorem): Let f be measurable with

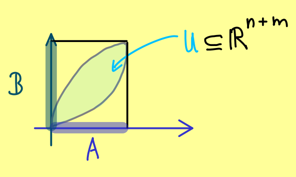
either $f : A \times B \longrightarrow [0, \infty]$ $\quad (A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m)$

or $f : A \times B \longrightarrow \mathbb{R}$ with $\int_{A \times B} |f| \, d\lambda^{(n+m)} < \infty$.

Then:

$$\int_{A \times B} f \, d\lambda^{(n+m)} = \int_A \left(\int_B f(x,y) \, d^m y \right) d^n x = \int_B \left(\int_A f(x,y) \, d^n x \right) d^m y$$

Problem:



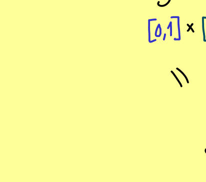
$f : U \longrightarrow \mathbb{R}$

$\hookrightarrow \tilde{f} : A \times B \longrightarrow \mathbb{R}$

$$(x,y) \mapsto \begin{cases} f(x,y) & \text{if } (x,y) \in U \\ 0 & \text{if } (x,y) \notin U \end{cases}$$

$$\int_U f \, d\lambda^{(n+m)} = \int_{A \times B} \tilde{f} \, d\lambda^{(n+m)} \overset{\text{Fubini}}{=} \int_A \left(\int_B \tilde{f}(x,y) \, d^m y \right) d^n x$$

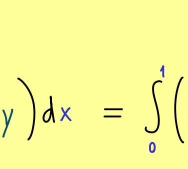
Example:



domain picture

area

volume



graph picture

$$\int_U 1 \, d(x,y)$$

$$\| $$

$$\int_{[0,1] \times [1,3]} \tilde{f}(x,y) \, d(x,y) \quad \text{with} \quad \tilde{f}(x,y) := \begin{cases} 1, & x \in [0,1], y \in [1, 3-x^2] \\ 0 & \text{else} \end{cases}$$

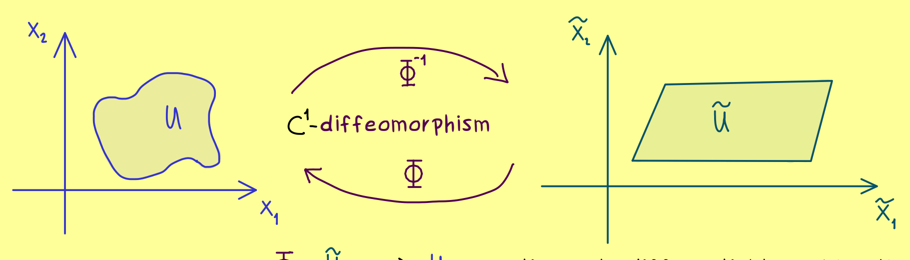
$$\overset{\text{Fubini}}{=} \int_0^1 \left(\int_1^3 \tilde{f}(x,y) \, dy \right) dx = \int_0^1 \left(\int_1^{3-x^2} 1 \, dy \right) dx$$

$$= \int_0^1 (3 - x^2 - 1) \, dx = \int_0^1 (2 - x^2) \, dx = \frac{5}{3}$$

Multidimensional Integration - Part 5

$f: U \longrightarrow \mathbb{R}$ measurable , $U \subseteq \mathbb{R}^n$ open

$\longrightarrow \int_U f(x)\, d^n x$



$\Phi : \tilde{U} \longrightarrow U$ continuously differentiable + bijective

$\Phi^{-1} : U \longrightarrow \tilde{U}$ continuously differentiable

<u>substitution:</u> $x = \Phi(\tilde{x}) \xrightarrow{\text{in one dimension!}} dx = \Phi'(\tilde{x})\, d\tilde{x}$

now!

$$d^n x = \left|\det\left(J_\Phi(\tilde{x})\right)\right| d^n \tilde{x}$$

<u>Change of variables formula:</u> $$\int_{\Phi[\tilde{U}]} f(x)\, d^n x = \int_{\tilde{U}} f(\Phi(\tilde{x})) \left|\det\left(J_\Phi(\tilde{x})\right)\right| d^n \tilde{x}$$

If one exists,
then also the other!

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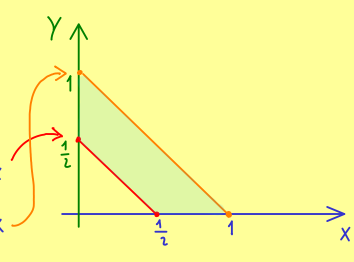
Multidimensional Integration - Part 6

Change of variables formula: $$\int_{\Phi[\tilde{U}]} f(x)\, d^n x = \int_{\tilde{U}} f(\Phi(\tilde{x}))\, \left|\det(J_\Phi(\tilde{x}))\right| \, d^n \tilde{x}$$

Example: $$\int_G \cos\left(\frac{x-y}{x+y}\right) d(x,y) \quad \text{with} \quad G := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \tfrac{1}{2} - x \leq \tfrac{1}{2} \leq x+y \leq 1 \leq 1+y \right\}$$

Sketch of G:

$$\tfrac{1}{2} - x \leq \tfrac{1}{2} \iff x \geq 0$$
$$1 \leq 1+y \iff y \geq 0$$
$$\tfrac{1}{2} \leq x+y \iff y \geq \tfrac{1}{2} - x$$
$$x+y \leq 1 \iff y \leq 1 - x$$



Diffeomorphism: $$\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ x+y \end{pmatrix}$$

$$J_\Phi(x,y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \det(J_\Phi(x,y)) = 2$$

Change of variables: $$\frac{1}{2} \cdot \int_G \cos\left(\frac{x-y}{x+y}\right) \cdot 2 \, d(x,y) = \frac{1}{2} \cdot \int_G f(\Phi(x,y)) \left|\det(J_\Phi(x,y))\right| d(x,y)$$

$$f(u,v) = \cos\left(\frac{u}{v}\right)$$

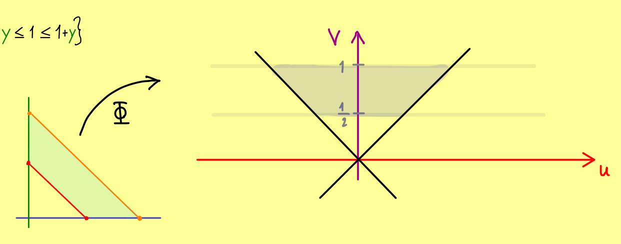
$$= \frac{1}{2} \cdot \int_{\Phi[G]} f(u,v) \, d(u,v)$$
$$= \frac{1}{2} \cdot \int_{\Phi[G]} \cos\left(\frac{u}{v}\right) d(u,v)$$

New integration domain: $$\Phi[G] = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; v \in \left[\tfrac{1}{2}, 1\right], \; u+v \geq 0, \; v-u \geq 0 \right\}$$
$$= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; v \in \left[\tfrac{1}{2}, 1\right], \; v \geq -u, \; v \geq u \right\}$$

$$\boxed{\begin{aligned} u &= x - y \\ v &= x + y \end{aligned}}$$

$G := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \tfrac{1}{2} - x \leq \tfrac{1}{2} \leq x+y \leq 1 \leq 1+y \right\}$

$$\boxed{\begin{aligned} u + v &= 2x \\ v - u &= 2y \end{aligned}}$$



Result: $$\int_G \cos\left(\frac{x-y}{x+y}\right) d(x,y) = \frac{1}{2} \cdot \int_{\Phi[G]} \cos\left(\frac{u}{v}\right) d(u,v)$$

$$\overset{\text{Fubini}}{=} \frac{1}{2} \int_{\frac{1}{2}}^{1} \Bigg(\underbrace{\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du}_{v \cdot \sin\left(\frac{u}{v}\right)\Big|_{u=-v}^{u=v}} \Bigg) dv = \frac{1}{2} \int_{\frac{1}{2}}^{1} v \cdot \Big(\sin(1) \underbrace{- \sin(-1)}_{+\sin(1)}\Big) dv$$

$$= \frac{1}{2} \cdot 2 \cdot \sin(1) \cdot \frac{1}{2} v^2 \Big|_{\frac{1}{2}}^{1} = \underline{\frac{3}{8} \sin(1)}$$