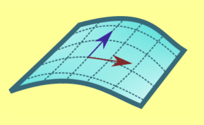
The Bright Side of Mathematics



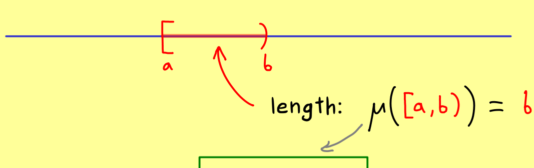
Multidimensional Integration - Part 1

Measure Theory → Multidimensional Integration ← Multivariable Calculus

Multidimensional Integration → Manifolds

- Definitions for integrals
- Change-of-variables formula
- Fubini's theorem
  - ↳ (Fubini-Tonelli theorem)
- Explicit calculations

Lebesgue measure on $\mathbb{R}$:



$a \leq b$

length: $\mu([a,b)) = b - a$

pre-measure

Carathéodory's Extension Theorem

$\varphi: \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty]$ outer measure

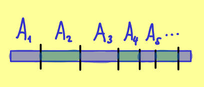
$\mathcal{L}(\mathbb{R}) = \mathcal{A}_\varphi$ σ-algebra of Lebesgue-measurable sets

$\lambda: \mathcal{L}(\mathbb{R}) \longrightarrow [0,\infty]$ Lebesgue measure on $\mathbb{R}$

measure

Properties:

- $\lambda(\emptyset) = 0$
- $\lambda\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \lambda(A_j)$ for $A_j \in \mathcal{L}(\mathbb{R})$, $A_j \cap A_i = \emptyset$ for $i \neq j$



- $\mathcal{L}(\mathbb{R})$ is larger than the Borel σ-algebra.
- If $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) = 0$, (A is called null set) then each $B \subseteq A$ satisfies $B \in \mathcal{L}(\mathbb{R})$.
- $\lambda([a,b)) = b - a$, $b \geq a$
- $\lambda(x + A) = \lambda(A)$ for all $x \in \mathbb{R}$, $A \in \mathcal{L}(\mathbb{R})$

(translation-invariant)

Definition (Lebesgue integral):

$$\int_A f \, d\lambda = \int_A f(x) \, d\lambda(x) = \int_A f(x) \, dx$$

defined by approximation with simple functions: