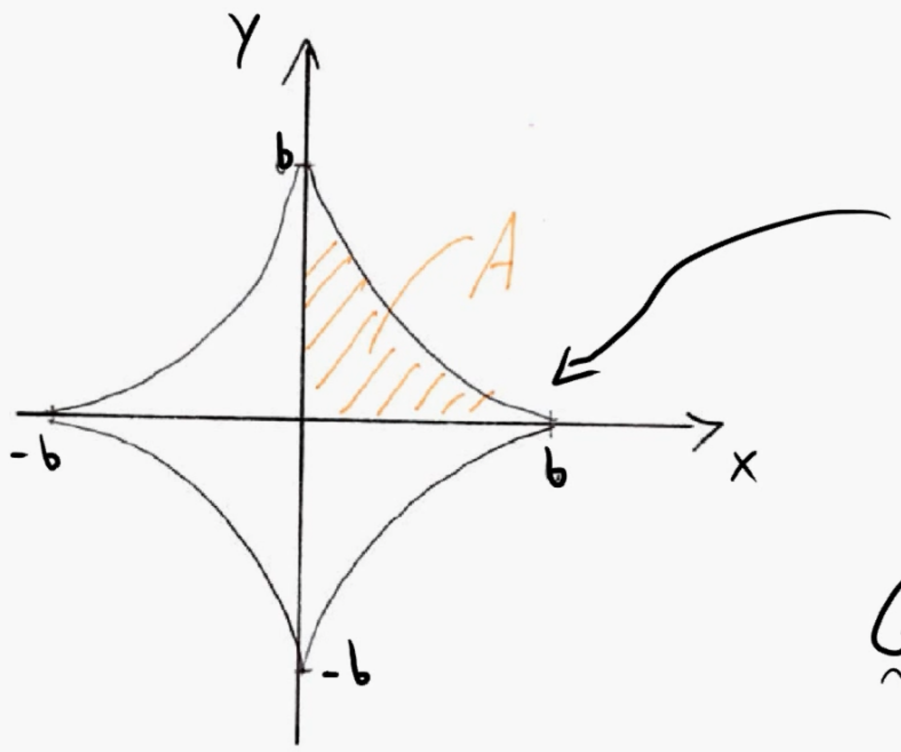
Aufgabe 9

Skizzieren Sie die Astroide beschrieben durch $|x|^{2/3} + |y|^{2/3} = a > 0$. Berechnen Sie den eingeschlossenen Flächeninhalt mit Hilfe von neuen Koordinaten $x = \rho \cos^3(\theta)$ und $y = \rho \sin^3(\theta)$.

$a = b^{2/3}$, $\boxed{b := a^{3/2}}$

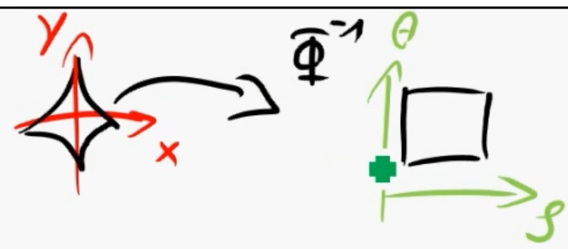


Stern Kurve hat Parametrisierung:

$$\theta \mapsto \begin{pmatrix} b\cos^3\theta \\ b\sin^3\theta \end{pmatrix}$$

Gesucht: Fläche $= 4 \cdot A = 4 \cdot \int_A 1 \, d(x,y)$

Neue Koordinaten: $\Phi(\rho,\theta) = \begin{pmatrix} \rho\cos^3\theta \\ \rho\sin^3\theta \end{pmatrix} \begin{matrix} = x \\ = y \end{matrix}$



Transformationsformel: $\displaystyle\int_{A = \Phi(B)} 1 \, d(x,y) = \int_B 1 \cdot \underbrace{|\det J_\Phi(\rho,\theta)|}_{?} \, d(\rho,\theta)$

Jacobi-Determinante:

$$\det J_\Phi(\rho,\theta) = \det \begin{pmatrix} \cos^3\theta & 3\rho\cos^2\theta\,(-\sin\theta) \\ \sin^3\theta & 3\rho\sin^2\theta\cos\theta \end{pmatrix} =$$

$$= \cos^4\theta\sin^2\theta \cdot 3\rho + \cos^2\theta\sin^4\theta \cdot 3\rho = 3\rho\cos^2\theta\sin^2\theta\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}$$

$$= 3\rho\,(\underbrace{\cos\theta\sin\theta}_{\frac{1}{2}\sin(2\theta)})^2 = 3\rho \cdot \frac{1}{4}\sin^2(2\theta) = \frac{3}{4}\rho \cdot \frac{1}{2}(1-\cos(4\theta))$$

$\left[\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2 x\right]$

$$= \frac{3}{8}\rho\,(1-\cos(4\theta))$$

Was ist B?

$$\Phi^{-1}(A) = B = \left\{ (\rho, \theta) \mid \rho \in [0, b] , \theta \in [0, \tfrac{\pi}{2}] \right\}$$

$$\text{Fläche} = 4 \cdot A = 4 \cdot \int_{\Phi(B)} 1 \, d(x,y) = 4 \cdot \int_{B} 1 \cdot \rho \frac{3}{8} (1 - \cos(4\theta)) \, d(\rho, \theta)$$

$$= 4 \cdot \int_0^{\pi/2} \left(\int_0^b \frac{3}{8} \rho \, (1 - \cos(4\theta)) \, d\rho \right) d\theta \Leftarrow \left(b^2 = a^3 \right)$$

$$= 4 \cdot \frac{3}{32} \pi a^3 = \underline{\frac{3}{8} \pi a^3}$$