Aufgabe 4 mehrdimensionale Integration (Satz v. Fubini)

(a) Skizziere die Menge

$$B := \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + 4y^2,\ x^2 + y^2 \leq 1\}\,.$$

(b) Berechne das zweidimensionale Integral

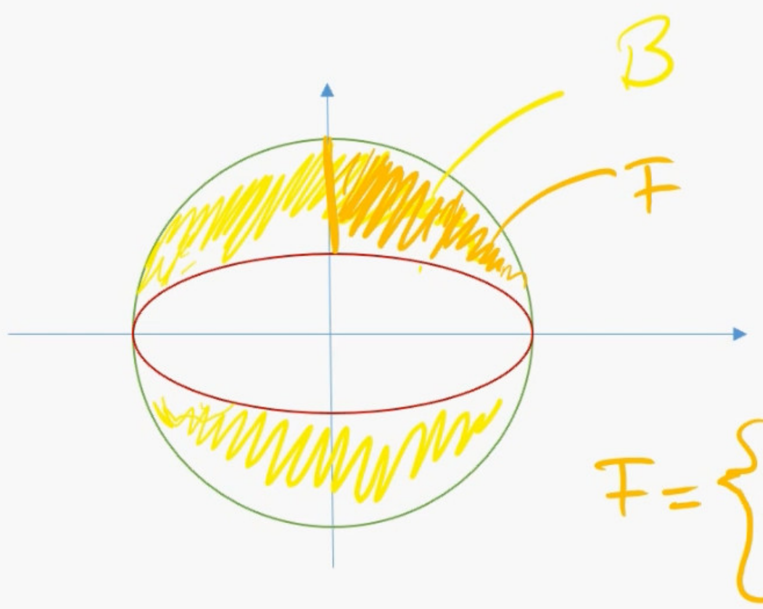
$$I = \int_B (|x| + |y|)\, d(x,y)$$

mit Hilfe des Satzes von Fubini auf zwei verschiedene Weisen.

$$B := \{(x,y) \in \mathbb{R}^2 \mid 1 \leq \underbrace{x^2 + 4y^2}_{\text{Ellipse}},\ \underbrace{x^2 + y^2 \leq 1}_{\text{Kreis mit Radius 1}}\}\,.$$

$$I = \int_B (|x| + |y|)\, d(x,y)$$

$(y \leq \sqrt{1-x^2})$



$$I = \int_B (|x|+|y|)\, d(x,y)$$

$$= 4 \cdot \int_F (x+y)\, d(x,y)$$

$$F = \left\{ (x,y) \,\middle|\, x \in [0,1],\ y \in \left[\tfrac{1}{2}\sqrt{1-x^2},\ \sqrt{1-x^2}\right] \right\}$$

$$F = \left\{ (x,y) \,\middle|\, y \in [0,1],\ x \in \left[\sqrt{1-4y^2},\ \sqrt{1-y^2}\right] \right\}$$

$$I = \int_B (|x| + |y|)\, d(x,y) = 4 \cdot \int_F (\underbrace{|x| + |y|}_{x \text{ und } y \geq 0})\, d(x,y)$$

$$= 4 \cdot \int_F (x + y)\, d(x,y) = 4 \cdot \int_0^1 \left(\int_{\frac{1}{2}\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x+y)\, dy \right) dx$$

$$= 4 \cdot \int_0^1 \left[x \cdot y + \frac{1}{2} y^2 \right]_{y = \frac{1}{2}\sqrt{1-x^2}}^{y = \sqrt{1-x^2}} dx$$

$$= 4 \cdot \int_0^1 \left[x\sqrt{1-x^2} + \frac{1}{2}(1-x^2) - x \cdot \frac{1}{2}\sqrt{1-x^2} - \frac{1}{2} \cdot \frac{1}{4}(1-x^2) \right] dx$$

$$= \underline{4 \cdot \frac{5}{12}}$$

$$I = \int_B (|x| + |y|)\, d(x,y) = 4 \int_F (x + y)\, d(x,y)$$

$$= 4 \cdot \int_0^1 \left(\int_{\sqrt{1-4y^2}}^{\sqrt{1-y^2}} (x+y)\, dx \right) dy$$

$$= 4 \cdot \int_0^1 \left[\frac{1}{2} x^2 + x \cdot y \right]_{x = \sqrt{1-4y^2}}^{x = \sqrt{1-y^2}} dy$$

$$= \underline{4 \cdot \frac{5}{12}}$$