Mehrdimensionale Integration

Aufgabe 2

(a) Skizziere die Menge

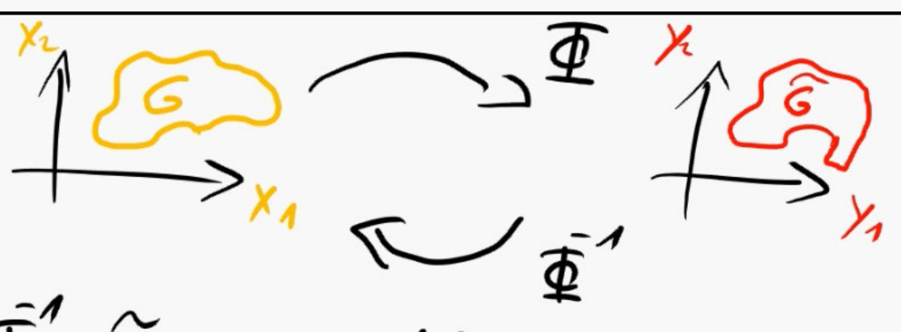
$$B := \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{1}{2} - x \leq \frac{1}{2} \leq x + y \leq 1 \leq 1 + y\right\}.$$

(b) Berechne das zweidimensionale Integral

$$I = \int_B \cos\left(\frac{x-y}{x+y}\right) \, d(x,y)$$

Hinweis: Verwenden Sie die Substitution $u = x - y$ und $v = x + y$ und damit die Transformationsformel.

Substitution (Transformationsformel)



$G, \tilde{G} \subseteq \mathbb{R}^n$ (offen), $\Phi : G \to \tilde{G}$

differenzierbar, bijektiv, und $\Phi^{-1} : \tilde{G} \to G$ differenzierbar.

Dann gilt für integrierbares f:

$$\int_{\Phi(G) = \tilde{G}} f(y_1, \dots, y_n) \, d(y_1, \dots, y_n) = \int_G \underbrace{f(\Phi(x_1, \dots, x_n)) \, |\det J_\Phi(x_1, \dots, x_n)|}_{\text{Funktion wird besser!}} \, d(x_1, \dots, x_n)$$

Merke: " $y = \Phi(x)$, $dy = \underbrace{\Phi'(x)}_{|\det J_\Phi(x)|} \, dx$ "

G wird "besser".

(b) Berechne das zweidimensionale Integral

$$I = \int_{\cancel{B}\,\tilde{G}} \cos\left(\frac{x-y}{x+y}\right) \, d(x,y)$$

Hinweis: Verwenden Sie die Substitution $u = x - y$ und $v = x + y$ und damit die Transformationsformel.

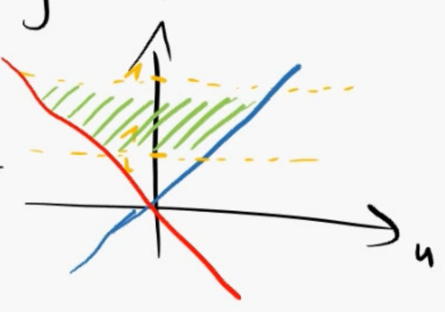
$\cancel{B}\,\tilde{G} := \left\{(x,y) \in \mathbb{R}^2 \mid \frac{1}{2} - x \leq \frac{1}{2} \leq x + y \leq 1 \leq 1 + y\right\}$

$\boxed{B = \tilde{G}}$

$$\Phi^{-1}(x,y) = \begin{pmatrix} x - y \\ x + y \end{pmatrix} \begin{matrix} = u \\ = v \end{matrix} \quad \Rightarrow \quad \begin{matrix} u + v = 2x \\ v - u = 2y \end{matrix}$$

$$\underset{G \,=}{\Phi^{-1}(\tilde{G})} = \left\{(u,v) \mid v \in [\tfrac{1}{2}, 1], \; u + v \geq 0, \; v - u \geq 0\right\}$$

$$= \Big\{(u,v) \mid v \in [\tfrac{1}{2}, 1], \; \underbrace{v \geq -u, \; v \geq u}_{u \in [-v, v]}\Big\}, \quad G =$$



Rechnung:

$$I = \int_{B = \tilde{G} = \Phi(G)} f(x,y) \, d(x,y) \overset{\text{Trans.}}{=} \int_G \underbrace{f(\Phi(u,v))}_{\cos\left(\frac{u}{v}\right)} \, |\det J_\Phi(u,v)| \, d(u,v)$$

$$\Phi(u,v) = \begin{pmatrix} \frac{1}{2}(u+v) \\ \frac{1}{2}(v-u) \end{pmatrix}$$

$$J_\Phi(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \det J_\Phi = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$= \int_G \cos\left(\frac{u}{v}\right) \cdot \frac{1}{2} \, d(u,v)$$

$$\overset{\text{Fubini}}{=} \frac{1}{2} \left[\int_{1/2}^{1} \left(\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du \right) dv \right] = \frac{1}{2} \int_{1/2}^{1} v \cdot \sin\left(\frac{u}{v}\right) \Big|_{u=-v}^{v} \, dv$$

$$= \frac{1}{2} \left[\int_{1/2}^{1} v \cdot \big(\sin(1) \underbrace{- \sin(-1)}_{+\sin(1)}^{\text{ung. Fkt}}\big) \, dv \right] = \frac{1}{2} \cdot 2 \sin(1) \left[\frac{1}{2} v^2 \right]_{\frac{1}{2}}^{1} = \frac{3}{8} \sin(1)$$