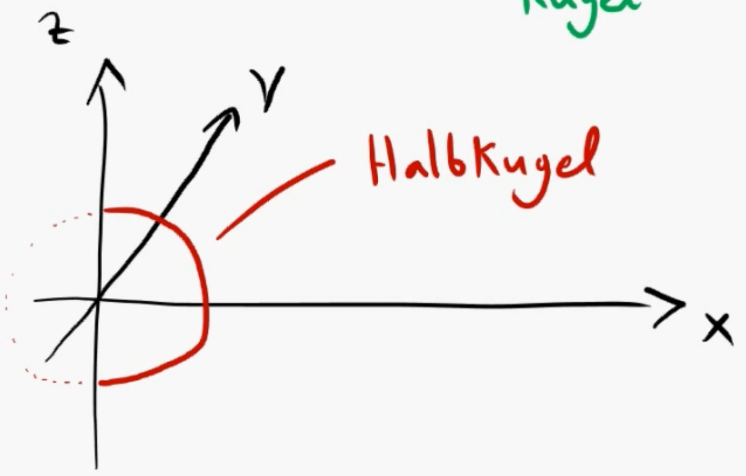
Aufgabe 8 Volumenberechnung mit mehrdim. Integralen

Skizzieren Sie den folgenden Körper und berechnen Sie das Volumen mit Hilfe von dreidimensionalen Integralen

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{\overbrace{x^2+y^2+z^2}^{r^2} \le 1}_{\text{Kugel}},\ \underbrace{\overset{r\sin\theta\sin\varphi}{y} \ge 0}_{\text{schneidet Kugel ab}}\}.$$

$$\mathrm{vol}(A) = \int_A 1 \, d(x,y,z)$$



A = Halbkugel, $\quad \mathrm{vol}(A) = \int_A 1 \, d(x,y,z)$

Transformation in Kugelkoordinaten

$$\underset{(x,y,z)}{A} \xrightarrow{\Phi^{-1}} \underset{(r,\varphi,\theta)}{\boxed{B}}$$

$$\Phi(r,\varphi,\theta) = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} \begin{array}{l} x \\ y \\ z \end{array}, \qquad \Phi^{-1} = ?$$

Transformationsformel:

$$\int_{\Phi(B)=A} \underbrace{f(x,y,z)}_{1} \, d(x,y,z) = \int_{\underset{r^2?}{B}} \underbrace{f(\Phi(r,\varphi,\theta))}_{1} \underbrace{|\det J_\Phi(r,\varphi,\theta)|}_{?} \, d(r,\varphi,\theta)$$

$$B = \Phi^{-1}(A) = \{(r,\varphi,\theta) \in \mathbb{R}^3 \mid \overbrace{r \in [0,\infty),\ \varphi \in [0,2\pi],\ \theta \in [0,\pi]}^{\text{immer bei Kugelkoordinaten}},\ r \le 1,\ r\sin\theta\sin\varphi \ge 0\}$$

$$B = \left\{ (r,\varphi,\theta) \in \mathbb{R}^3 \;\middle|\; r \in [0,1],\ \varphi \in [0,2\pi],\ \theta \in [0,\pi],\ \underbrace{r}_{\geq 0}\,\underbrace{\sin\theta}_{\geq 0}\,\underbrace{\sin\varphi}_{\Downarrow\ \varphi\in[0,\pi]} \geq 0 \right\}$$

$$= \left\{ (r,\varphi,\theta) \in \mathbb{R}^3 \;\middle|\; r \in [0,1],\ \varphi \in [0,\pi],\ \theta \in [0,\pi] \right\}$$

$$\det J_{\Phi}(r,\varphi,\theta) = r^2 \sin\theta$$ (merken, nachrechnen!)

$$\mathrm{vol}(A) = \int_A 1 \, d(x,y,z) \overset{\text{Transf.}}{=} \int_B 1 \, |r^2 \sin\theta| \, d(r,\varphi,\theta)$$

$$\overset{\text{Fubini}}{=} \int_0^1 \left(\int_0^\pi \left(\int_0^\pi r^2 \sin\theta \, d\theta \right) d\varphi \right) dr = \underline{\frac{4}{6}\pi}$$