BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

Measure Theory – Part 23

Carathéodory's Extension Theorem: X set, $\mathcal{A} \subseteq P(X)$ semiring of sets

$\mu : \mathcal{A} \longrightarrow [0, \infty]$ pre-measure.

Then: μ has an extension $\tilde{\mu} : \sigma(\mathcal{A}) \longrightarrow [0,\infty]$ as a measure!

Proof: Define outer measure: $\varphi : P(X) \longrightarrow [0,\infty]$

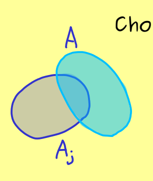
$$\varphi(B) := \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) \;\middle|\; A_j \in \mathcal{A} \text{ with } B \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

(see part 21)

First question: $\mathcal{A} \subseteq \mathcal{A}_\varphi$ ← φ-measurable sets

Proof of first question: Choose $A \in \mathcal{A}$, $Q \in P(X)$, $\varphi(Q) < \infty$.

Choose $A_j \in \mathcal{A}$ with $Q \subseteq \bigcup_{j=1}^{\infty} A_j$. Then:



$$\sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \mu\big((A_j \cap A) \cup (A_j \setminus A)\big) = \underbrace{\sum_{j=1}^{\infty} \mu(A_j \cap A)}_{\geq \varphi(Q \cap A)} + \underbrace{\sum_{j=1}^{\infty} \mu(A_j \setminus A)}_{\geq \varphi(Q \cap A^c)}$$

inf on both sides

$$\Longrightarrow \varphi(Q) \geq \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

Second question: $\varphi(A) = \mu(A)$ for all $A \in \mathcal{A}$

Proof of second question: $\varphi(A) \leq \mu(A)$ ✓ (by infimum definition)

property of pre-measure: $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ ← disjoint

$$\Longrightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(A_j)$$

σ-subadditivity: $A \subseteq \bigcup_{j=1}^{\infty} A_j \Longrightarrow \mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$

infimum on right-hand side

$$\Longrightarrow \mu(A) \leq \varphi(A) \checkmark$$

Recall important proposition for outer measures:

For an outer measure φ, we find a measure $\tilde{\mu} : \mathcal{A}_\varphi \longrightarrow [0,\infty]$

with $\tilde{\mu}(A) = \varphi(A) = \mu(A)$ for all $A \in \mathcal{A}$.