The Bright Side of Mathematics

Outer Measures – Part 3

Important Proposition: For an outer measure $\varphi: \mathcal{P}(X) \longrightarrow [0,\infty]$:

(a) $\mathcal{A}_\varphi := \{ A \subseteq X \mid A \ \ \varphi\text{-measurable} \}$ is a σ-algebra

(b) $\mu: \mathcal{A}_\varphi \longrightarrow [0,\infty]$ with $\mu(A) := \varphi(A)$ is a measure

Proof:

- $\emptyset \in \mathcal{A}_\varphi$? Is $\emptyset$ φ-measurable? $\varphi(Q) = \underbrace{\varphi(\underbrace{Q \cap \emptyset}_{\emptyset})}_{=0} + \varphi(\underbrace{Q \cap \emptyset^c}_{Q})$ ✓

- $X \in \mathcal{A}_\varphi$? Is X φ-measurable? $\varphi(Q) = \varphi(Q \cap X) + \varphi(Q \cap \underbrace{X^c}_{=\emptyset})$ ✓

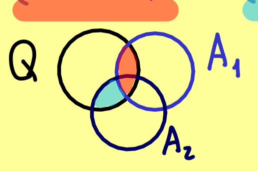
- $A \in \mathcal{A}_\varphi \implies \varphi(Q) = \varphi(Q \cap A) + \varphi(Q \cap A^c) = \varphi(Q \cap A^c) + \varphi(Q \cap (A^c)^c)$

  $\implies A^c \in \mathcal{A}_\varphi$ ✓

- union with two sets: $A_1, A_2 \in \mathcal{A}_\varphi$

  $$\varphi(Q) = \varphi(Q \cap A_1) + \varphi(\overbrace{Q \cap A_1^c}^{\tilde{Q}}) = \varphi(Q \cap A_1) + \varphi(\tilde{Q} \cap A_2) + \varphi(\tilde{Q} \cap A_2^c)$$

  $$\geq \varphi\big((Q \cap A_1) \cup (\tilde{Q} \cap A_2) \big) + \varphi(\underbrace{\tilde{Q} \cap A_2^c}_{Q \cap (A_1^c \cap A_2^c)})$$



  $$Q \cap (A_1^c \cap A_2^c) \overset{\parallel}{} \; Q \cap (A_1 \cup A_2)^c$$

  $\implies \varphi(Q) \geq \varphi\big(Q \cap (A_1 \cup A_2)\big) + \varphi\big(Q \cap (A_1 \cup A_2)^c\big)$

  $\implies A_1 \cup A_2 \in \mathcal{A}_\varphi$ ✓

- countable union: $A_1, A_2, A_3, \ldots \in \mathcal{A}_\varphi$, pairwise disjoint! $\qquad A := \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}_\varphi$?

  $$\varphi(\overset{\hat{Q} \cap (A_1 \cup A_2)}{\overset{\parallel}{Q}}) = \varphi(\overset{\hat{Q} \cap (A_1 \cup A_2)}{\overset{\parallel}{Q}} \cap A_1) + \varphi(\overset{\hat{Q} \cap (A_1 \cup A_2)}{\overset{\parallel}{Q}} \cap A_1^c) = \varphi(\hat{Q} \cap A_1) + \varphi(\hat{Q} \cap A_2)$$

  induction: $\varphi\Big(\hat{Q} \cap \bigcup_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \varphi(\hat{Q} \cap A_j) \qquad (*)$

  we have: $$\varphi(\hat{Q}) = \underbrace{\varphi\Big(\hat{Q} \cap \bigcup_{j=1}^{n} A_j\Big)}_{\overset{(*)}{=} \sum_{j=1}^{n} \varphi(\hat{Q} \cap A_j)} + \underbrace{\varphi\Big(\hat{Q} \cap \Big(\bigcup_{j=1}^{n} A_j\Big)^c\Big)}_{\geq \varphi(\hat{Q} \cap A^c) \text{ (monotonicity)}}$$

  $$\overset{n \to \infty}{\implies} \varphi(\hat{Q}) \geq \sum_{j=1}^{\infty} \varphi(\hat{Q} \cap A_j) + \varphi(\hat{Q} \cap A^c) \overset{\sigma\text{-subadditivity}}{\geq} \varphi(\hat{Q} \cap A) + \varphi(\hat{Q} \cap A^c) \geq \varphi(\hat{Q})$$

  $\implies A \in \mathcal{A}_\varphi$ $\square$