The Bright Side of Mathematics

Outer measures - part 2 : examples

$\varphi : \mathcal{P}(X) \longrightarrow [0,\infty]$ with
- $\varphi(\emptyset) = 0$
- monotonic
- σ-subadditive

Examples: (1) $\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty]$, $\varphi(A) := \begin{cases} 0 & , A = \emptyset \\ 1 & , A \neq \emptyset \end{cases}$

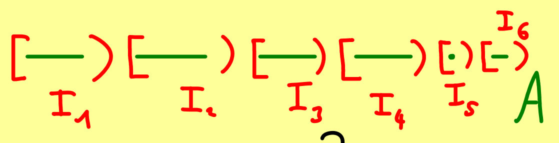
$\rightsquigarrow$ outer measure but <u>not</u> a measure!

(2) $\varphi : \mathcal{P}(\mathbb{N}) \longrightarrow [0,\infty]$, $\varphi(A) := \begin{cases} |A| & , A \text{ finite} \\ \infty & , A \text{ not finite} \end{cases}$

$\rightsquigarrow$ outer measure but <u>also</u> a measure! (counting measure)

(3) $\mathcal{I} = \{ [a,b) \mid a,b \in \mathbb{R},\ a \leq b \}$, $\mu([a,b)) = b-a$ ("length")

Define $\varphi : \mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty)$ by :



$$\varphi(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) \;\middle|\; I_j \in \mathcal{I},\ A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

$\rightsquigarrow$ φ is an outer measure!

Proof: (a) $\varphi(\emptyset) = 0$ ✓

(b) monotonicity: $A \subseteq B \Rightarrow \varphi(B) = \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) \;\middle|\; I_j \in \mathcal{I},\ \underset{\underset{A}{\cup\!\!|}}{B} \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$

$$\geq \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) \;\middle|\; I_j \in \mathcal{I},\ A \subseteq \bigcup_{j=1}^{\infty} I_j \right\} = \varphi(A)$$

(c) Show: $\varphi\left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \varphi(A_n)$

Proof of this: Let $\varepsilon > 0$. Choose $\varepsilon_n > 0$ with $\sum_{n \in \mathbb{N}} \varepsilon_n = \varepsilon$.

Then there are intervals $I_{j,n}$ with:

$\varphi(A_n) \geq \sum_{j=1}^{\infty} \mu(I_{j,n}) - \varepsilon_n$ and $A_n \subseteq \bigcup_{j=1}^{\infty} I_{j,n}$

Then: $\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} I_{j,n} = \bigcup_{j,n} I_{j,n}$

$$\Rightarrow \varphi\left(\bigcup_{n \in \mathbb{N}} A_n \right) \overset{(b)}{\leq} \varphi\left(\bigcup_{j,n} I_{j,n} \right) \leq \sum_{j,n} \mu(I_{j,n}) = \sum_{n \in \mathbb{N}} \left(\sum_{j \in \mathbb{N}} \mu(I_{j,n}) \right)$$

$$\leq \sum_{n \in \mathbb{N}} \left(\varphi(A_n) + \varepsilon_n \right) = \sum_{n \in \mathbb{N}} \varphi(A_n) + \varepsilon$$