



## Measure theory - Part 2

Definition:  $\mathcal{A} \subseteq \mathcal{P}(X)$  is called a  $\sigma$ -algebra:

$$(a) \quad \emptyset, X \in \mathcal{A}$$

$$(b) \quad A \in \mathcal{A} \Rightarrow A^c := X \setminus A \in \mathcal{A}$$

$$(c) \quad A_i \in \mathcal{A}, i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

$A \in \mathcal{A}$  is called an  $\mathcal{A}$ -measurable set.

Easy to show:  $\mathcal{A}_i$   $\sigma$ -algebra on  $X$ ,  $i \in I$  (index set)

Then  $\bigcap_{i \in I} \mathcal{A}_i$  is also a  $\sigma$ -algebra on  $X$ .

Definition: For  $\mathcal{M} \subseteq \mathcal{P}(X)$ , there is a smallest  $\sigma$ -algebra that contains  $\mathcal{M}$ :

$$\sigma(\mathcal{M}) := \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{M} \\ \mathcal{A} \text{ } \sigma\text{-algebra}}} \mathcal{A} \quad \leftarrow \text{ } \sigma\text{-algebra generated by } \mathcal{M}$$

Example:  $X = \{a, b, c, d\}$ ,  $\mathcal{M} = \{\{a\}, \{b\}\}$ .

$$\sigma(\mathcal{M}) = \left\{ \emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{c, d\} \right\}$$

Definition: Let  $(X, \mathcal{T})$  be a topological space. } We need  
(Let  $X$  be a metric space.) "open sets".  
(Let  $X$  be a subset of  $\mathbb{R}^n$ )

$\mathcal{B}(X)$  Borel  $\sigma$ -algebra on  $X$ .

(the  $\sigma$ -algebra generated by the open sets)

$$\mathcal{B}(X) := \sigma(\mathcal{T})$$