Measure theory – Part 2

Definition: $\mathcal{A} \subseteq P(X)$ is called a σ-algebra :

(a) $\emptyset, X \in \mathcal{A}$

(b) $A \in \mathcal{A} \Rightarrow A^c := X \setminus A \in \mathcal{A}$

(c) $A_i \in \mathcal{A}, \; i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

$A \in \mathcal{A}$ is called an $\mathcal{A}$-measurable set.

Easy to show: $\mathcal{A}_i$ σ-algebra on X, $i \in I$ (index set)

Then $\bigcap_{i \in I} \mathcal{A}_i$ is also a σ-algebra on X.

Definition: For $\mathcal{M} \subseteq P(X)$, there is a smallest σ-algebra that contains $\mathcal{M}$:

$$\sigma(\mathcal{M}) := \bigcap_{\substack{\mathcal{A} \supseteq \mathcal{M} \\ \mathcal{A} \text{ σ-algebras}}} \mathcal{A}$$

← σ-algebra generated by $\mathcal{M}$

Example: $X = \{a, b, c, d\}$, $\mathcal{M} = \{\{a\}, \{b\}\}$.

$$\sigma(\mathcal{M}) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}, \{a,c,d\}, \{c,d\}\}$$

Definition: Let $(X, \mathcal{T})$ be a topological space.
(Let X be a metric space.)
(Let X be a subset of $\mathbb{R}^n$)

$\Big\}$ We need "open sets".

$\mathcal{B}(X)$ Borel σ-algebra on X.

(the σ-algebra generated by the open sets)

$$\mathcal{B}(X) := \sigma(\mathcal{T})$$