

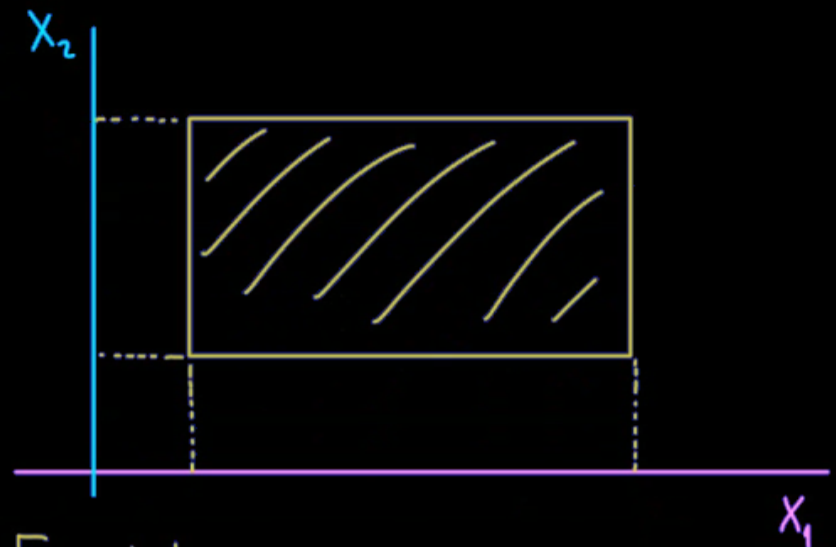
Measure Theory - Part 19

Fubini's Theorem

$(X_1, \mathcal{A}_1, \mu_1)$ measure space

$(X_2, \mathcal{A}_2, \mu_2)$ measure space

$\leadsto (X_1 \times X_2, \mathcal{A}, \mu)$ product measure



$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$$

product measure is uniquely determined if μ_1, μ_2 are σ -finite

Fubini's theorem: Let μ_1, μ_2 be σ -finite, μ be the product measure and

$f: X_1 \times X_2 \rightarrow [0, \infty]$ measurable. [or $f \in L^1(\mu)$]

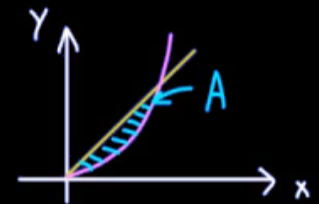
$$\text{Then: } \int_{X_1 \times X_2} f \, d\mu = \int_{X_2} \left(\int_{X_1} f(x, y) \, d\mu_1(x) \right) d\mu_2(y) = \int_{X_1} \left(\int_{X_2} f(x, y) \, d\mu_2(y) \right) d\mu_1(x)$$

Example: μ Lebesgue measure for \mathbb{R}^2

$$\int_A f \, d\mu = ?$$

$$A = \{(x, y) \in [0, 1] \times [0, 1] \mid x \geq y \geq x^2\}$$

$$f(x, y) = 2 \cdot x \cdot y$$



$$\int_A f \, d\mu = \int_{\mathbb{R}^2} f \cdot \chi_A \, d\mu = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \chi_A(x, y) \, dy \right) dx$$

$$= \int_0^1 \left(\int_{x^2}^x 2x \cdot y \, dy \right) dx = 2 \int_0^1 x \left(\int_{x^2}^x y \, dy \right) dx$$

antiderivative: $\frac{1}{2}y^2$

$$= 2 \cdot \int_0^1 x \cdot \frac{1}{2} (x^2 - x^4) \, dx = \int_0^1 (x^3 - x^5) \, dx = \frac{1}{12}$$