



Measure Theory - Part 14

Radon-Nikodym theorem and Lebesgue's decomposition theorem

$(X, \mathcal{A}, \lambda)$ measure space
 $\begin{matrix} \parallel \\ \mathbb{R} \end{matrix}$ $\begin{matrix} \parallel \\ \mathcal{B}(\mathbb{R}) \end{matrix}$ $\begin{matrix} \parallel \\ \lambda \end{matrix}$
 (Lebesgue measure) (reference measure) ($\lambda([a, b]) = b - a$)

Another measure $\mu: \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$

Definitions: (a) μ is called absolutely continuous (w.r.t λ)

if $\lambda(A) = 0 \Rightarrow \mu(A) = 0$

for all $A \in \mathcal{B}(\mathbb{R})$. One writes: $\mu \ll \lambda$

(b) μ is called singular (w.r.t. λ)

if there is $N \in \mathcal{B}(\mathbb{R})$ with $\lambda(N) = 0$ and $\mu(N^c) = 0$.

One writes: $\mu \perp \lambda$

Example: δ_0 Dirac measure ($\delta_0(\{0\}) = 1$)

$\Rightarrow \delta_0 \perp \lambda$ (Choose $N = \{0\}$)

Theorem: $\mu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ (σ -finite)

(a)
Lebesgue's
decomposition
theorem

There are two measures (uniquely determined)

$$\mu_{ac}, \mu_s : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$$

$$\text{With } \mu = \mu_{ac} + \mu_s, \quad \mu_{ac} \ll \lambda, \quad \mu_s \perp \lambda.$$

(b)
Radon-
Nikodym
theorem

There is a measurable map $h : \mathbb{R} \rightarrow [0, \infty)$

$$\text{With } \mu_{ac}(A) = \int_A h \, d\lambda$$

density

for all $A \in \mathcal{B}(\mathbb{R})$.