The Bright Side of Mathematics

Measure Theory – Part 14

Radon-Nikodym theorem and Lebesgue's decomposition theorem

$(X, \mathcal{A}, \lambda)$ measure space, $X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, $\lambda = \lambda$ (Lebesgue measure) — reference measure $\big(\lambda([a,b]) = b - a\big)$

Another measure $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0, \infty]$

Definitions: (a) μ is called absolutely continuous (w.r.t λ)

if $\lambda(A) = 0 \implies \mu(A) = 0$

for all $A \in \mathcal{B}(\mathbb{R})$.

One writes: $\mu \ll \lambda$

(b) μ is called singular (w.r.t. λ)

if there is $N \in \mathcal{B}(\mathbb{R})$ with $\lambda(N) = 0$ and $\mu(N^c) = 0$. ($N^c = \mathbb{R} \setminus N$)

One writes: $\mu \perp \lambda$

Example: δ_0 Dirac measure $\big(\delta_0(\{0\}) = 1\big)$

$\implies \delta_0 \perp \lambda$ (Choose $N = \{0\}$)

Theorem: $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$ (σ-finite)

Lebesgue's decomposition theorem

$\boxed{(a)}$ There are two measures (uniquely determined)

$$\mu_{ac}, \mu_s : \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$$

with $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} \ll \lambda$, $\mu_s \perp \lambda$.

Radon-Nikodym theorem

$\boxed{(b)}$ There is a measurable map $h : \mathbb{R} \to [0,\infty)$ with

$$\mu_{ac}(A) = \int_A \underbrace{h}_{\text{density}} \, d\lambda$$

for all $A \in \mathcal{B}(\mathbb{R})$.