BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Measure Theory – Part 9

Fatou's Lemma

$(X, \mathcal{A}, \mu)$ measure space

$f_n : X \to [0, \infty]$ measurable for all $n \in \mathbb{N}$

$$\int_X \underbrace{\liminf_{n\to\infty} f_n}\, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$$

$\liminf_{n\to\infty} f_n : X \to [0, \infty]$

$$g(x) := \Big(\liminf_{n\to\infty} f_n\Big)(x) := \lim_{n\to\infty} \Big(\overbrace{\inf_{k\geq n} f_k(x)}^{g_n(x)} \Big) \in [0, \infty]$$

measurable

$g_1 \leq g_2 \leq g_3 \leq \cdots$ (monotonically increasing)

Proof:

$$\int_X \lim_{n\to\infty} g_n \, d\mu \overset{\text{mon. conv. theorem}}{=} \lim_{n\to\infty} \int_X g_n \, d\mu = \liminf_{n\to\infty} \int_X g_n \, d\mu$$

We know: $\int_X g_n \, d\mu \leq \int_X f_n \, d\mu$ for all $n \in \mathbb{N}$

$$\Rightarrow \int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu \qquad \square$$