BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Measure Theory – Part 18

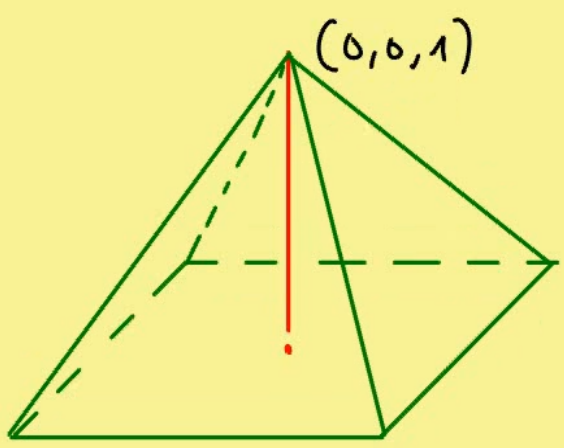
Cavalieri's principle – An example

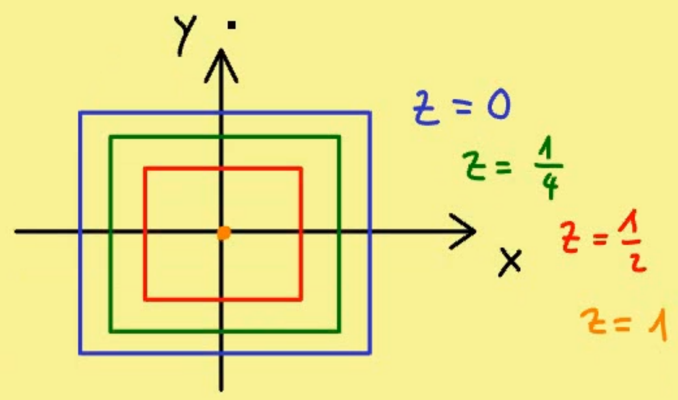
Calculate the volume of the pyramid with corners

$(-1,-1,0)$, $(-1,1,0)$, $(1,-1,0)$, $(1,1,0)$, $(0,0,1)$. $(K \subset \mathbb{R}^3)$

↳ Lebesgue measure in $\mathbb{R}^3$: μ

(product measure construction with Lebesgue measure on $\mathbb{R}$)





$$K = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 0 \leq z \leq 1 \;,\; |x| \leq 1-z \;,\; |y| \leq 1-z \right\}$$

μ product measure of μ_1 and μ_2. → Lebesgue measure on $\mathbb{R}^2$ (x- and y-coordinate)

↳ Lebesgue measure in $\mathbb{R}$ (z-coordinate)

$$\mu(K) = \int_{\mathbb{R}} \underbrace{\mu_2(M_{z_0})}_{=(2\cdot(1-z_0))^2} \, d\mu_1(z_0) \quad , \quad M_{z_0} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; |x| \leq 1-z_0 \,,\, |y| \leq 1-z_0 \right\}$$

(area of the square)

only for $z_0 \in [0,1]$

$$= \int_{[0,1]} 4\cdot(1-z_0)^2 \, d\mu_1(z_0) = \int_0^1 4(1-z_0)^2 \, dz_0 = 4\cdot\left(-\frac{1}{3}(1-z_0)^3\right)\Big|_0^1 = \underline{\frac{4}{3}}$$