The Bright Side of Mathematics

Measure Theory – Part 17

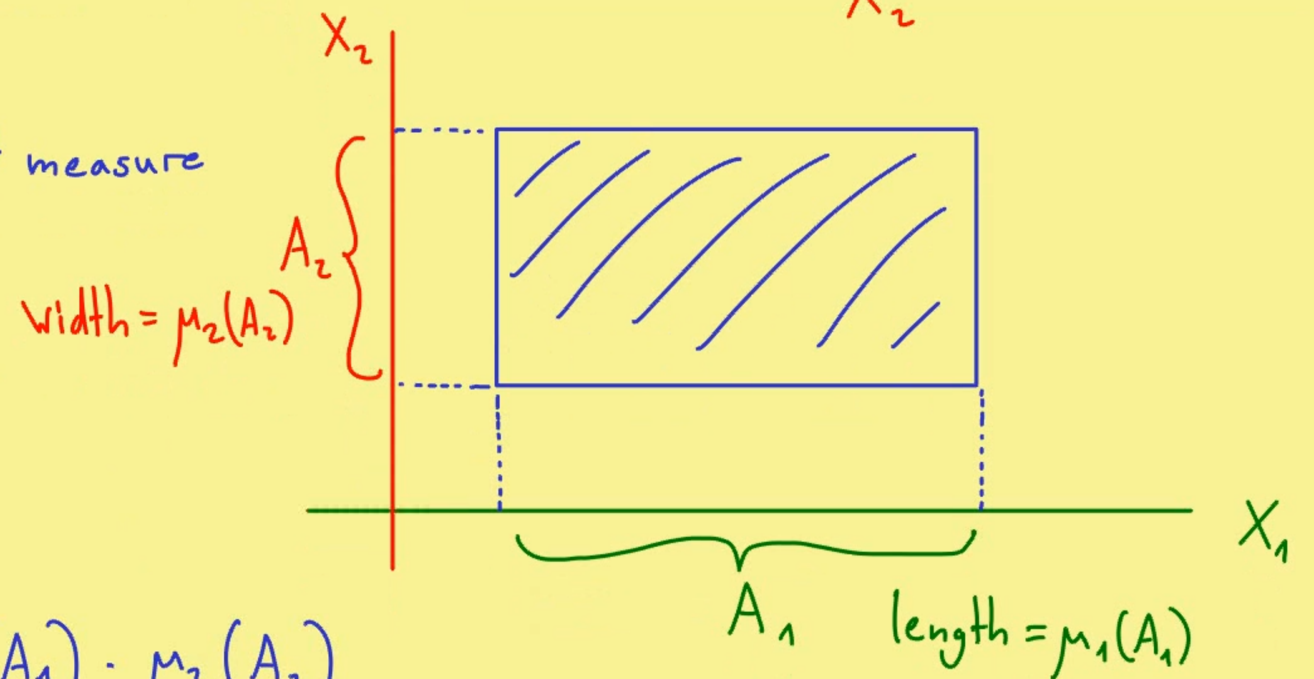
Product measure and Cavalieri's principle

$(X_1, \mathcal{A}_1, \mu_1)$ measure space

$(X_2, \mathcal{A}_2, \mu_2)$ measure space

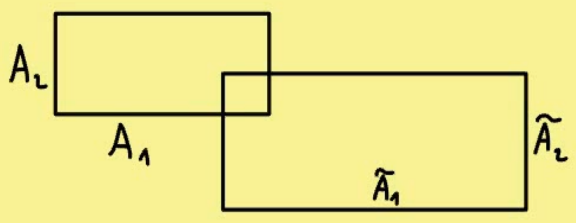
$\rightsquigarrow$

$(X_1 \times X_2, \mathcal{A}, \mu)$ product measure



$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$$

Product σ-algebra: $\mathcal{A} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$



Set of rectangles $(= \mathcal{A}_1 \times \mathcal{A}_2)$ are not a σ-algebra (but a semiring of sets)

Product measure: Define μ as $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ for all $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, and use Carathéodory's extension theorem.

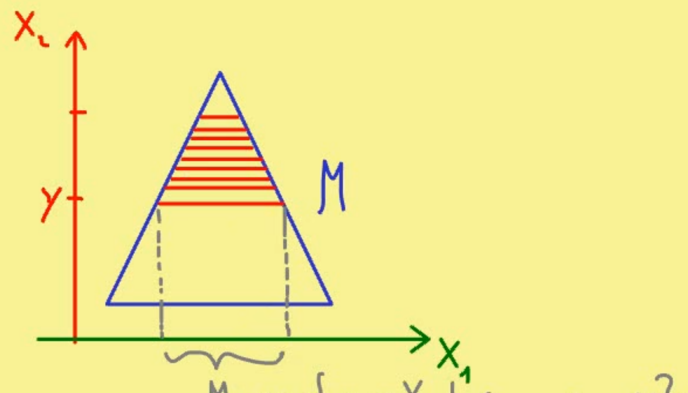
(Product measure in general not unique)

Proposition: If μ_1, μ_2 are σ-finite, then there is exactly one measure μ with $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$.

It satisfies:

$$\mu(M) = \int_{X_2} \mu_1(M_y)\, d\mu_2(y)$$

$$= \int_{X_1} \mu_2(M_x)\, d\mu_1(x)$$



$M_y := \{ x_1 \in X_1 \mid (x_1, y) \in M \}$

$M_x := \{ x_2 \in X_2 \mid (x, x_2) \in M \}$

[Cavalieri's principle]